Bipolar Problem of Space – Charge in Dielectric. I. System without Trapping

by

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Summary. A model of a dielectric with bipolar injection of charge in the form of a planeparallel condenser is presented. Space and time distributions of charge and induction as well as time courses of the resorption current were determined. The resorption state is described by partial differential equations. Initial conditions were determined on the basis of the stationary state of absorption. The theory of partial quasi – linear differential equations was used in the analytical and numerical considerations.

1. Introduction. One of the fundamental problems [1–4] in the macroscopic theory of space charge in dielectrics is that of relating the time characteristics of absorption currents $j_a(t)$ or resorption currents $j_r(t)$ to space and time distributions of space – charge density $q_v(x, t)$ inside the dielectric. Considered in this context [5] is the so called initial problem, i.e. the determination of $j(t)$ on the basis of the given initial distribution $q_v(x, 0)$ and given boundary conditions, and also two reverse problems—the determination of $q_v(x, 0)$ from the measured course of current density $j_{(t)}$ and boundary conditions, and the identification of boundary conditions also on the basis of the known course of current density $j_{(t)}$. The general solution of the above mentioned problems is extremely complex and requires a number of simplifications concerning first of all the physical model of current (omission of various components, e.g. diffusion current) and also of polarization mechanisms [6] represented in the so – called material equation and by

the assumed sign of the charge accumulated in the given dielectric.
Practically the entire space – charge theory so far evolved is based on the
so called monopolar approximation, i.e. the admittance of the distribution
$q_v(x, t)$ of only one sign. There are known, however, cases of e.g. time
courses of resorption current which till now could not be satisfactorily
explained on the premises of the existing monopolar theory. The emerged
problem can be solved in various ways—for example one could introduce
additional, suitably chosen, components of electric displacement or appro-
priately modify the material coefficients appearing in expressions describing
current components. It appears, however, that the most natural way of
solving the problem is the introduction of the bipolar concept of space –
– charge distribution. This solution arises from the relevant tenets of solid
state physics. The present paper and the subsequent works related to it
will be devoted to the bipolar concept of space – charge. Our aim is to
work out a suitable theory without judging, for the time being, the
adequacy of either of the concepts. In the present paper some of the
possibilities of solving the bipolar problem of resorption current admitting
the existence of two charge distributions—the positive $^+q_v(x, t)$ and the
negative $^-q_v(x, t)$—are considered from the mathematical point of view.

2. Basic equations. The monopolar problem [1] is given in the system
of planeparallel condenser with the following fundamental equations:
— the continuity equation

$$\frac{\partial}{\partial x}(\mu g_v E) = -\frac{\partial g_v}{\partial t},$$

— the Poisson equation

$$\frac{\partial D}{\partial x} = q_v,$$

— the material equation

$$D = \varepsilon_0 \varepsilon E,$$

— the boundary condition

$$\int_0^L E dx = 0.$$

The bipolar problem containing an additional unknown requires the
introduction of a fifth equation describing the dependence between the
distributions $^+q_v(x, t)$ and $^-q_v(x, t)$. We assume here the law of mass
action expressed by the relation

$$^+q_v(x, t)\,^-q_v(x, t) = \text{const.}$$

The following simplifications were assumed in this paper:
— in the continuity equation the diffusion current and the so called ohm current σE resulting from the motion of compensated free carriers are omitted;
— in the material equation only the follow – up polarization leading to the simple vector relation $D = \varepsilon_0 \, \varepsilon E$ is allowed;
— the electron – hole mechanism of conductance is allowed only (the others, such as that of ionic current carriers, are omitted).
Under such conditions the basic equations take the form

$$(1) \qquad \frac{\partial}{\partial x} \{\mu_p \, p\,(x,\,t)\, E\,(x,\,t) + \mu_n \, n\,(x,\,t)\, E\,(x,\,t)\} + \frac{\partial p\,(x,\,t)}{\partial t} - \frac{\partial n\,(x,\,t)}{\partial t} = 0$$

$$(2) \qquad \frac{\partial D\,(x,\,t)}{\partial x} = e_0 \, \{(p\,(x,\,t) - p_0) - (n\,(x,\,t) - n_0)\},$$

$$(3) \qquad n\,(x,\,t)\, p\,(x,\,t) = n_0 \, p_0 = K$$

$$(4) \qquad D\,(x,\,t) = \varepsilon_0 \, \varepsilon E\,(x,\,t)$$

with the boundary condition

$$(5) \qquad \int\limits_0^L E\,(x,\,t)\, dx = 0,$$

where: $p,\, n$—nonequilibrium concentrations of holes and electrons, respectively; $p_0,\, n_0$—equilibrium concentrations of holes and electrons, respectively; $\mu_p,\, \mu_n$—mobility of holes and electrons, respectively; D—electric induction; E—electric field intensity; L—length of sample; ε—relative dielectric permeability; K—constant; $e_0 = 1.6 \cdot 10^{-19}$ C; $\varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m.
Equations (1)–(5) describe the transient state of the resorption current. The density of resorption current $j\,(t)$ is determined from (1) and (2).

$$(6) \qquad j_{(t)} = \frac{\partial D\,(x,\,t)}{\partial t} + e_0 \, \mu_p \, p\,(x,\,t)\, E\,(x,\,t) + e_0 \, \mu_n \, n\,(x,\,t)\, E\,(x,\,t)$$

The aim of the present considerations is the determination of the time course of density $j_{(t)}$, of the space – time distributions of charges $^+q_v\,(x,\,t) = e_0 \, p\,(x,\,t)$ and $^-q_v\,(x,\,t) = -e_0 \, n\,(x,\,t)$, and of the space – time distributions of induction $D\,(x,\,t)$. In terms of the systematics suggested by Badian, this corresponds to the solution of the initial problem.
The analytical solution of the resorption current problem is fairly complicated for different mobilities $\mu_n \neq \mu_p$ of electrons and holes. The problem becomes a good deal simpler for equal mobilities $\mu_n = \mu_p$; this

situation was assumed in the first stage of study. The form of the assumed equations requires information about the initial conditions. We will determine the initial values of charge on the basis of the absorption current's steady state [7]. The steady state is described by the equations

$$(7) \qquad e_0 \, \mu_p \, p\,(x) \, E\,(x) + e_0 \, \mu_n \, n\,(x) \, E\,(x) = j_a = \text{const.},$$

$$(8) \qquad \frac{dD\,(x)}{dx} = e_0 \, \{(p\,(x) - p_0) - (n/x) - n_0)\},$$

$$(9) \qquad n\,(x)\,p\,(x) = K,$$

$$(10) \qquad D\,(x) = \varepsilon_0 \, \varepsilon E\,(x).$$

with the boundary condition

$$(11) \qquad \int_0^L E\,(x)\,dx = U, \quad U = \text{const.}, \quad U > 0.$$

The initial values of charges ^+q_v and ^-q_v are described by the equations

$$(12) \qquad ^+q_v\,(x, 0) = e_0 \, p\,(x) \qquad ^-q_v\,(x, 0) = -e_0 \, n\,(x).$$

The above way of determining initial values remains valid through – out the subsequent considerations.

3. Problem solution. A partial solution of the resorption current problem may be obtained analytically assuming equal mobilities $\mu_n = \mu_p$. Given this assumption it is possible to determine the distribution of induction $D\,(x, t)$ as well as distributions of charges $^+q_v\,(x, t)$ and $^-q_v\,(x, t)$. The determination of the time course of current density $j_{(t)}$ and of the variation in time of charge in an arbitrary point of the dielectric is a rather complex problem. It has been solved numerically [8].

The present work is devoted to the analytical solution of the problem with the simplifying assumption $\mu_n = \mu_p = \mu$. In virtue of the assumed trapping – free state, equilibrium concentrations were taken as equal ($n_0 = p_0$) which allows to simplify the right – hand sides of equations (2) and (8). In what follows, the problem is considered in the system of dimensionless variables $^\pm Q_v$, x', D', t', j', U' described by the equations

$$(13) \qquad ^+Q_v = \frac{p}{K^{1/2}}; \quad ^-Q_v = \frac{-n}{K^{1/2}}, \quad x' = \frac{x}{L}, \quad D' = \frac{D}{e_0 \, LK^{1/2}},$$

$$t' = \frac{e_0 \, \mu K^{1/2}}{\varepsilon_0 \, \varepsilon} \, t, \quad j' = \frac{\varepsilon_0 \, \varepsilon}{e_0^2 \, \mu LK} \, j, \quad u' = \frac{\varepsilon_0 \, \varepsilon}{e_0 \, L^2 \, K^{1/2}} \, U.$$

Omitting the primes in the symbols, we can write the equations describing the transient state of resorption current in a system of normalized variables as follows

(14) $\qquad \dfrac{\partial\, {}^{+}Q_v\,(x,\,t)}{\partial t}+\dfrac{\partial\, {}^{-}Q_v\,(x,\,t)}{\partial t}+\dfrac{\partial}{\partial x}\,\{[{}^{+}Q_v\,(x,\,t)-{}^{-}Q_v\,(x,\,t)]\,D\,(x,\,t)\}=0,$

(15) $\qquad\qquad\qquad \dfrac{\partial D\,(x,\,t)}{\partial x}={}^{+}Q_v\,(x,\,t)+{}^{-}Q_v\,(x,\,t),$

(16) $\qquad\qquad\qquad {}^{+}Q_v\,(x,\,t)\,{}^{-}Q_v\,(x,\,t)=-1,$

with the boundary condition

(17) $\qquad\qquad\qquad \displaystyle\int_0^1 D\,(x,\,t)\,dx=0$

the current density $j_{(t)}$ being described by the formula

(18) $\qquad j_{(t)}=\dfrac{\partial D\,(x,\,t)}{\partial t}+[{}^{+}Q_v\,(x,\,t)-{}^{-}Q_v\,(x,\,t)]\,D\,(x,\,t)$

The aim of these considerations is the determination of the time course of current density $j_{(t)}$ and of the space – time distributions of induction $D\,(x,\,t)$ and charge ${}^{-}Q_v\,(x,\,t)$. We shall be making use of the equation

(19) $\qquad j_{(t)}=-\displaystyle\int_0^1 F\,\dfrac{(1-{}^{-}Q_v^2)}{{}^{-}Q_v^2}\,\dfrac{\partial\, {}^{-}Q_v}{\partial x}\,dx;\qquad F=\displaystyle\int_0^x D\,(x,\,t)\,dx.$

which results from (16)–(18). From equation (19) it is possible to infer about the change of current flow direction on the basis of the character of distribution of the negative charge ${}^{-}Q_v$.

We must at the same time consider the difference between the nonequilibrium value $({}^{-}Q_v\neq-1)$ and the equilibrium value ${}^{-}Q_v=-1$. In agreement with the assumptions, the initial values of charge ${}^{-}Q_v\,(x,\,0)$ were determined from the steady state of the absorption current which indicates that the initial values of ${}^{-}Q_v\,(x,\,0)$ may be described by the formula

(20) $\qquad\qquad\qquad {}^{-}Q_v\,(x,\,0)=-\dfrac{j_a+\sqrt{j_a^2-4D_u^2}}{2D_u}$

with the induction D_u satisfying the differential equation

(21) $\qquad\qquad\qquad D_u\,\dfrac{dD_u}{dx}=-\sqrt{j_a^2-4D_u^2};\quad D_u>0.$

The further considerations concerning the transient state of resorption current are based on the thus formulated way of determining initial values. Transformation of equations (14)–(16) gives partial quasilinear differential equations with respect to the charge ${}^{-}Q_v$

(22) $\qquad (1+{}^{-}Q_v^2)\,\dfrac{\partial\, {}^{-}Q_v}{\partial t}+D\,(1-{}^{-}Q_v^2)\,\dfrac{\partial\, {}^{-}Q_v}{\partial x}-(1+{}^{-}Q_v^2)\,(-1+{}^{-}Q_v^2)=0.$

Hence, in agreement with the theory of characteristics, we write the equations

$$(23) \qquad \frac{dx}{dt} = D \frac{1 - {}^-Q_v^2}{1 + {}^-Q_v^2}$$

and

$$(24) \qquad \frac{d\,{}^-Q_v}{dt} = -1 + {}^-Q_v^2.$$

In the subsequent considerations we assume the uniqueness of the solution of equation (23)

$$(25) \qquad x_{(t)} = \int_0^t D \frac{1 - {}^-Q_v^2}{1 + {}^-Q_v^2}\, dt + x\,(0).$$

From the uniqueness of the solution of equation (23) there results the inequality

$$(26) \qquad \left. \frac{dx_{(t)}}{dx_{(0)}} \right|_{t\,=\,\text{const.}} > 0$$

which plays a fundamental role in considerations concerning the space – time distributions of charge and induction.

From the assumed way of determining initial values it results that on the characteristic the charge ${}^-Q_{v(x,(t),t)}$ is always smaller than the equilibrium value ${}^-Q_v = -1$. It is infered from this [8] that there exists inside the dielectric a negative excess charge $\frac{\partial D}{\partial x} < 0$ and that through every point of the dielectric there passes the characteristic (25) always directed to electrode $x = 0$ or $x = 1$. In other words, it is not necessary to know the boundary conditions in order to determine the space – time distributions of charge and induction and the time course of resorption current. Smycz [9] has proved this for the monopolar charge problem. It is demonstrated in this work that the above statement can be generalized to include also the case of bipolar charge.

We now show a transformation which can be used to determine the monotonicity of the distribution of charge ${}^-Q_v$. The change of distribution of charge ${}^-Q_v(x, t)$ may be described as follows

$$(27) \qquad \left. \frac{\partial\,{}^-Q_v(x,t)}{\partial x} \right|_{x\,=\,x(t)} = \left. \frac{\partial\,{}^-Q_v(x,t)}{\partial x_{(0)}} \right|_{x\,=\,x_{(0)}} \cdot \left. \frac{1}{\dfrac{dx_{(t)}}{dx_{(0)}}} \right|_{t\,=\,\text{const.}}$$

Hence, on the basis of (20), (21), (24) and (26), it is inferred that at arbitrary fixed time $t > 0$ the charge distribution ^-Q_v is decreasing $\dfrac{\partial\,^-Q_v}{\partial x} < 0$. On the other hand, it is known that at an arbitrary point of the dielectric there exists a negative excess charge $\dfrac{\partial D}{\partial x} < 0$ and that the values of $^-Q_v(x, t)$ are always smaller than the equilibrium value $^-Q_v = -1$. Hence, on the basis of equation (19) it is concluded that the direction of the resorption current is always opposite to the direction of the absorption current in steady state; we express this by the relation

$$(28) \qquad\qquad j_{(t)} < 0; \quad t > 0.$$

The inequality (28) together with the obtained conclusion about the character of distribution of the negative charge $^-Q_v(x, t)$ enables a more exact determination of space – time distributions of induction $D(x, t)$.

The monotonic distribution of the negative charge $\dfrac{\partial\,^-Q_v}{\partial x} < 0$ and the identity

$$(29) \qquad \frac{\partial^2 D}{\partial x^2} = \frac{1 + \,^-Q_v^2}{^-Q_v^2}\,\frac{\partial\,^-Q_v}{\partial x}$$

determine the convexity of induction distribution, i.e.

$$(30) \qquad\qquad \frac{\partial^2 D}{\partial x^2} > 0.$$

In turn, the boundary condition (17) may be written in the form

$$(31) \qquad \int\limits_0^1 D\,dx = \int\limits_0^{x_0} D\,dx + \int\limits_{x_0}^1 D\,dx = 0$$

where x_0 denotes the zero point of induction $D(x_0, t)_{t = \text{const.}} = 0$. After a suitable transformation it is inferred that the point x_0 lies closer to the electrode $x = 1$ and that the values of induction next to the electrodes $D(0, t)$ and $D(1, t)$ satisfy the inequality

$$(32) \qquad 0 < D(0, t) < |D(1, t)|; \quad D(1, t) < 0,$$

In other words, in resorption state the zero point of induction lies closer to the electrode which in steady state of absorption current introduces majority carriers.

We now turn to the variation in time of inductions $D(0, 1)$ and $D(1, t)$ next to the electrodes. After substituting $x = 0$ in formula (18) we get

$$(33) \qquad j_{(t)} = \frac{\partial D(0, t)}{\partial t} + [^+Q_v(0, t) - \,^-Q_v(0, t)]\,D(0, t)$$

Hence, from (28), (32) and the positive value of the difference $^+Q_v - ^-Q_v$, it results that next to the electrode $x = 0$ the value of induction $D(0, t)$ constantly decreases $\dfrac{\partial D(0, t)}{\partial t} < 0$.

Similarly, for $x = 1$ we have

$$(34) \qquad j_{(t)} = \frac{\partial D(1, t)}{\partial t} + [^+Q_v(1, t) - ^-Q_v(1, t)] \, D(1, t).$$

In turn

$$(35) \qquad j_{(t)} = \int_0^1 [^+Q_v(x, t) - ^-Q_v(x, t)] \, D(x, t) \, dx =$$

$$= [^+Q_v(c, t) - ^-Q_v(c, t)] \, D(c, t) \qquad 0 < c < 1$$

which results from equations (17) and (18) and from the theorem of mean value for integrals. From the monotonicity of distributions $D(x, t)$ and $^-Q_v(x, t)$ and from (28) there results the inequality

$$(36) \qquad -j_{(t)} = [^+Q_v(c, t) - ^-Q_v(c, t)] \, |D(c, t)| <$$

$$< [^+Q_v(1, t) - ^-Q_v(1, t)] \, |D(1, t)|.$$

After taking into consideration (28), (32) and (35) it is found that the value of induction $D(1, t)$ constantly increases $\dfrac{\partial D(1, t)}{\partial t} > 0$.

The results of analytical considerations are illustrated in Figs. 1 and 2. The time course of current density $j_{(t)}$ and changes in time of charge

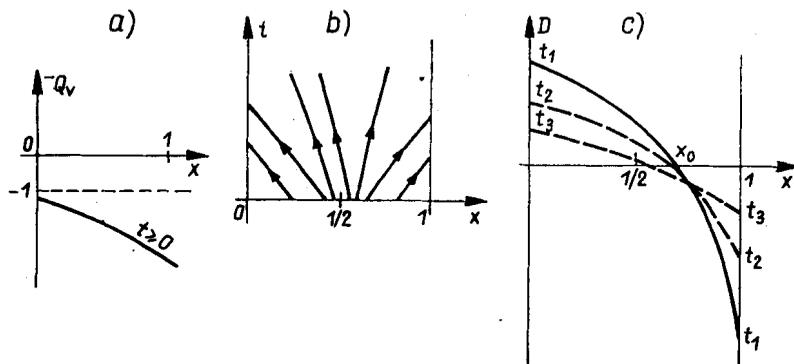

Fig. 1. Examples of resorption curves
a—distribution of negative charge at arbitrary time $t \geq 0$, b—characteristics, c—distribution of electric induction at different times $t_1 < t_2 < t_3$
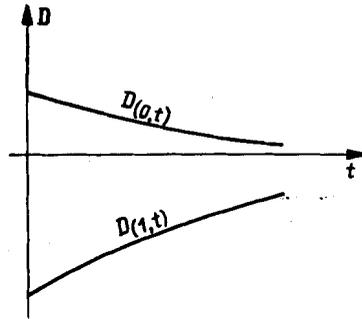
Fig. 2. Time course of induction $D(0, t)$ and $D(1, t)$ next to the electrodes
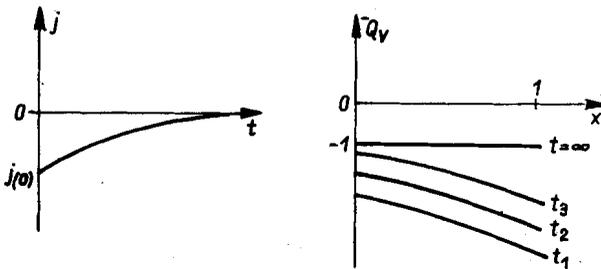

Fig. 3. Examples of resorption curves
a—time course of resorption current density, b—distribution of negative charge at different times $t_1 < t_2 < t_3$

$^-Q_v(x, t)$ in an arbitrary point of the dielectric were determined numerically [8]. The numerical calculations confirmed the properties presented in Figs. 1 and 2. The time course of $j_{(t)}$ is an increasing function with negative values. The charge ^-Q_v constantly increases at any point of the dielectric. Exemplary results obtained with a computer are presented in Fig. 3.

4. Conclusions. In determining the space – time distributions of electric charge and induction and the time course of resorption current density one does not have to know the boundary conditions. When the mobility of electrons and holes is equal there may appear in the entire dielectric a negative excess charge $\dfrac{\partial D}{\partial x} < 0$. In such conditions the time course of resorption current density is an increasing function with negative values. The distributions of concentrations $n(x, t)$ and $p(x, t)$ are monotonic in the entire dielectric and tend with time to equilibrium value $n_o = p_o$. The time courses of $D(0, t)$ and $D(L, t)$ of induction next to the electrodes are monotonic and of different absolute values. Electric induction is of negative sign at the electrode which in stationary state of absorption

current introduced majority carriers. The point of zeroing of induction distribution lies closer to the electrode which in stationary state of the absorption current introduced majority carriers. This point is mobile [8] and it constantly shifts in the direction of the electrode at which electric induction is of positive sign. The resorption current vanishes quicker in time [8] than the highest concentration of excess charge. As to the behaviour of monopolar charge injected into the dielectric, analysed in previously published works, it is worth stressing that the admission of bipolar distributions does not result in significant changes. This fact was not obvious in advance since the internal interactions of carriers of both signs might have demonstrated that the admission of only the law of mass action is somewhat arbitrary and that it is possible that the assumption of other mechanisms of electric conduction could lead to characteristics of $q_v(x, t)$ different in time. This problem will be the subject of further study.

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Б. Свистач, **Задача носителей обоих знаков, инжектированных в диэлектрик. Часть I. Диэлектрик без ловушек**

В настоящей статье описывается модельдвойной инжекции в диэлектрик в системе плоского конденсатора. Определяются пространственно-переходные характеристики заряда, электрической индукции и переходные характеристики тока резорбции. Состояние резорбции описывается дифференциальными уравнениями с частными производными. Начальные величины определяются по стационарному состоянию абсорбции. В аналитических и вычислительных-решениях применяется теория квазилинейных дифференциальных уравнений с частными производными.