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Nonrelativistic Change of Magnetic Flux Within a Closed Conductor; Translatory and Rotary Motion

by

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Summary. The equality

$$\frac{d}{dt}\iint\limits_{S(L)} \mathbf{B}d\mathbf{S} = \iint\limits_{S(L)} \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S} - \oint\limits_{L} (\mathbf{v}\times\mathbf{B})\, d\mathbf{L}$$

is proved in the case when contour L is rigid and performs a translatory and rotary motion around an axis in the plane of contour L.

1. Introduction. One of the fundamental equations of electrodynamics describes Faraday's induction law

$$\mathscr{E} = -\frac{d\Phi}{dt}, \tag{1}$$

where $\mathscr{E} = \oint\limits_{L} \mathbf{E}d\mathbf{L}$ — circulation of the electric field intensity vector $\mathbf{E}$, L — contour along the conductor, $\Phi = \iint\limits_{S(L)} \mathbf{B}d\mathbf{S}$ — flux of the magnetic induction vector $\mathbf{B}$ through an arbitrary smooth surface spread on the curve L, and t — time.

Electromotive force may appear in an immobile conductor in an inertial system as a result of variability in time vector $\mathbf{B}$. In such a case equation (1) takes the form

$$\mathscr{E} = \iint\limits_{S(L)} -\frac{\partial \mathbf{B}}{\partial t} d\mathbf{S}. \tag{2}$$

When a closed conductor travels with velocity $\mathbf{v}$ in a magnetic field with induction $\mathbf{B}$, then each unit of free charge in the conductor is acted upon by the Lorentz force $\mathbf{E}_L = \mathbf{v} \times \mathbf{B}$. In this case the induced electromotive force takes the value

$$\mathscr{E} = \oint_L (\mathbf{v} \times \mathbf{B})\, d\mathbf{L}. \tag{3}$$

If the conductor moves in a field of magnetic induction $\mathbf{B}$ which varies in time and space, then by superposition [3] of cases (2) and (3) we get

$$\mathscr{E} = \iint_{S(L)} -\frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{S} + \oint_L (\mathbf{v} \times \mathbf{B})\, d\mathbf{L}. \tag{4}$$

Hence, on the basis of (1), we get the differential-integral relation

$$\frac{d}{dt} \iint_{S(L)} \mathbf{B} d\mathbf{S} = \iint_{S(L)} \frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{S} - \oint_L (\mathbf{v} \times \mathbf{B})\, d\mathbf{L}. \tag{5}$$

Dependence (5) defines the manner of differentiating the surface integral when the integration range and the integrand vary in time. The reasoning leading to dependence (5) is presented in the literature [1] but this reasoning is not free from inaccuracies. In this paper we intend to give the assumptions and the proof of equation (5).

2. Problem solution. We begin by proving equality (5) in the case when curve L translates in a plane and in space and also performs rotary movement. The following assumptions are made here:

— curve L lies on the plane $0XY$ and is of constant shape,

— the plane region bounded by curve L is normal to axes $0X$ and $0Y$,

— the linear velocity $\mathbf{v}$ of an arbitrary point of the curve is many times smaller than velocity of light, $|\mathbf{v}| \ll c$,

— the considerations will concern the dextrorotary system $0XYZ$.

An additional assumption for equation (5) is that vector $\mathbf{B}$ lacks a source

$$\operatorname{div} \mathbf{B} = 0. \tag{6}$$

In a subsequent part of the paper we will show a case in which formula (5) is satisfied without assuming property (6).

We make use of the following theorem [2]

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t)\, dx = f(b(t); t)\, \frac{db}{dt} - f(a(t); t)\, \frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f(x; t)}{\partial t}\, dx. \tag{7}$$

Proceeding to formula (5), the following cases of motion of the closed curve L are considered:

1. Translatory motion on the plane $0XY$. At time $t = 0$ the plane region D_0 (Fig. 1) is described by the inequalities

(8) $$D_0 = \{(x_0; y_0);\ a_0 \leqslant x_0 \leqslant b_0;\ f_1(x_0) \leqslant y_0 \leqslant f_2(x_0)\}.$$

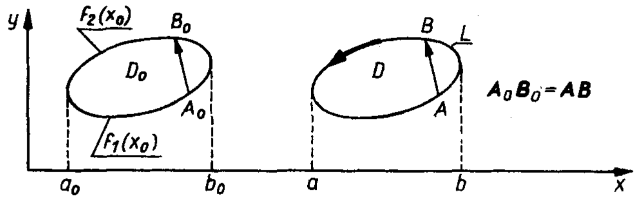

Fig. 1. Position of curve L at arbitrary time $t \geqslant 0$. Arrow indicates orientation of curve L

The leading vector $\mathbf{r} = (x; y; 0)$ and its derivative $\dfrac{d\mathbf{r}}{dt} = \mathbf{v} = (v_x; v_y; 0)$ define the position and velocity of an arbitrary point of curve L:

(9) $$\frac{dx}{dt} = v_x;\qquad \frac{dy}{dt} = v_y$$
$$x = \int_0^t v_x\,dt + x_0;\qquad y = \int_0^t v_y\,dt + y_0.$$

From these one obtains the initial values x_0 and y_0

(10) $$x_0 = x - S_x;\qquad y_0 = y - S_y$$

where

(11) $$S_x = \int_0^t v_x\,dt;\qquad S_y = \int_0^t v_y\,dt.$$

Since at time $t = 0$ $y_0 = f(x_0)$, where f denotes one of the functions f_1 or f_2, at time $t > 0$ there is satisfied the equality

(12) $$y = S_y + f(x - S_x).$$

Hence, on the basis of (8), the region $D_{(L)}$ (Fig. 1) is described by the inequalities

(13) $$D_{(L)} = \{(x; y):\ a_{(t)} \leqslant x \leqslant b_{(t)};\ S_y + f_1(x - S_x) \leqslant y \leqslant S_y + f_2(x - S_x)\}.$$

It is further assumed that the magnetic induction vector has coordinates $\mathbf{B} = (B_x; B_y; B_z)$ in the system $0XYZ$. The system $0XYZ$ is dextrorotary. The flux of vector $\mathbf{B}$ through the surface $D(L)$ spread on curve L is thus of the form

(14) $$\Phi = \iint_{D(L)} \mathbf{B}d\mathbf{S} = \iint_{D(L)} B_z\,dxdy.$$

The following additional denotations are introduced

$$\varphi_1 = S_y + f_1(x - S_x); \qquad \varphi_2 = S_y + f_2(x - S_x).$$

Hence, in view of (13), the double integral (14) becomes the iterated integral

$$\Phi = \int_a^b dx \int_{\varphi_1}^{\varphi_2} B_z(x; y; t)\, dy; \qquad z = 0. \tag{15}$$

Applying twice the theorem (7) to the integral (15) we get

$$\frac{d\Phi}{dt} = \frac{db}{dt} \int_{\varphi_1(b,t)}^{\varphi_2(b,t)} B_z(b; y; t)\, dy - \frac{da}{dt} \int_{\varphi_1(a,t)}^{\varphi_2(a,t)} B_z(a; y; t)\, dy + \int_a^b dx \frac{\partial}{\partial t} \int_{\varphi_1(x,t)}^{\varphi_2(x,t)} B_z(x; y; t)\, dy \tag{16}$$

$$\frac{\partial}{\partial t} \int_{\varphi_1(x,t)}^{\varphi_2(x,t)} B_z(x; y; t)\, dy = B_z(x; \varphi_2; t) \left(\frac{dS_y}{dt} - \frac{\partial \varphi_2}{\partial x} \cdot \frac{dS_x}{dt} \right) - B_z(x; \varphi_1; t) \left(\frac{dS_y}{dt} - \frac{\partial \varphi_1}{\partial x} \cdot \frac{dS_x}{dt} \right) + \int_{\varphi_1(x,t)}^{\varphi_2(x,t)} \frac{\partial B_z(x; y; t)}{\partial t}\, dy. \tag{17}$$

In (17) use was made of the condition of rigidity of curve L, i.e. $S_x = S_x(t)$ and $S_y = S_y(t)$. From relations (11) and from the rigidity of curve L it results that

$$\frac{da}{dt} = \frac{db}{dt} = \frac{dS_x}{dt} = v_x; \qquad \frac{dS_y}{dt} = v_y. \tag{18}$$

After taking into consideration (18) in formula (17) and arranging terms, the derivative of function $\Phi(t)$ takes the form

$$\frac{d\Phi}{dt} = \iint_{D(L)} \frac{\partial B_z}{\partial t}\, dx\, dy + \int_a^b v_x B_z(x; \varphi_1; t) \frac{\partial \varphi_1}{\partial x}\, dx + \int_{\varphi_1(b,t)}^{\varphi_2(b,t)} v_x B_z(b; y; t)\, dy + \int_b^a v_x B_z(x; \varphi_2; t) \frac{\partial \varphi_2}{\partial x}\, dx + \tag{19}$$

(19) (cont.)
$$+\int_{\varphi_2(a,t)}^{\varphi_1(a,t)} v_x B_z(a; y: t)\,dy + \int_a^b -v_y B_z(x; \varphi_1; t)\,dx + \int_b^a -v_y B_z(x; \varphi_2; t)\,dx$$

that is

(20)
$$\frac{d\Phi}{dt} = \iint_{D(L)} \frac{\partial B_z}{\partial t}\,dx\,dy + \oint_L v_x B_z\,dy - v_y B_z\,dx$$

or the equivalent form

(21)
$$\frac{d\Phi}{dt} = \iint_{D(L)} \frac{\partial B_z}{\partial t}\,dx\,dy - \oint_L \begin{vmatrix} dx & dy & 0 \\ v_x & v_y & 0 \\ B_x & B_y & B_z \end{vmatrix} = \iint_{D(L)} \frac{\partial \mathbf{B}}{\partial t}\,d\mathbf{S} - \oint_L (\mathbf{v}\times\mathbf{B})\,d\mathbf{L}.$$

Since function $\Phi(t)$ is defined by the integral (14), we have

$$\frac{d}{dt}\iint_{D(L)} \mathbf{B}\,d\mathbf{S} = \iint_{D(L)} \frac{\partial \mathbf{B}}{\partial t}\,d\mathbf{S} - \oint_L (\mathbf{v}\times\mathbf{B})\,d\mathbf{L}$$

which proves the correctness of formula (5) in the case of the plane region $S(L) = D(L)$. Note that condition (6) was not used in the above considerations. It will be used in considerations of spatial motion.

2. *Translatory motion in space.* In this case we must assume a suitable system of coordinates. Here we assume that at time $t = 0$ the curve L lies on the plane $0XY$. According to the definition of translatory motion, at time $t > 0$ curve L lies on a plane parallel to the plane $0XY$. Since the leading vector $\mathbf{r} = (x, y, z)$, the velocity vector $\mathbf{v} = d\mathbf{r}/dt$ has three coordinates $\mathbf{v} = (v_x, v_y, v_z)$ and then relations (9) must be complemented with the dependences

(22)
$$v_z = \frac{dz}{dt}; \qquad z = \int_0^t v_z\,dt; \qquad z(0) = 0.$$

In this case the integrand in the integral (15) must be replaced by the function $B_z = B_z(x, y, z(t), t)$ and formula (20) by

(23)
$$\frac{d\Phi}{dt} = \iint_{D_{xy}} \left(\frac{\partial B_z}{\partial z}\cdot v_z + \frac{\partial B_z}{\partial t}\right) dx\,dy + \oint_L v_x B_z\,dy - v_y B_z\,dx$$

where D_{xy} — projection of plane surface $S(L)$ spread on curve L onto the plane $0XY$. From the rigidity of curve L ($\mathbf{v} = \text{const.}$ on curve L) and from condition (6) it results that

$$(24)\qquad \iint\limits_{D_{xy}} \frac{\partial B_z}{\partial z}\cdot v_z\,dx\,dy = v_z \iint\limits_{D_{xy}} -\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)dx\,dy.$$

On the basis of Green's theorem, the integral (24) becomes the curvilinear integral

$$(25)\qquad v_z \iint\limits_{D_{xy}} -\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)dx\,dy = -v_z \oint\limits_{L'} -B_y\,dx + B_x\,dy$$

where L' — projection of curve L onto the plane $0XY$. Note that the circular integral (25) over curve L' is equal to the circular integral over curve L. It thus results from (25) and (23) that

$$(26)\qquad \frac{d\Phi}{dt} = \iint\limits_{D_{xy}} \frac{\partial B_z}{\partial t}\,dx\,dy + \oint\limits_{L} (v_z B_y - v_y B_z)\,dx + (v_x B_z - v_z B_x)\,dy$$

or equivalently that

$$(27)\qquad \frac{d\Phi}{dt} = \iint\limits_{D_{xy}} \frac{\partial B_z}{\partial t}\,dx\,dy - \oint\limits_{L} \begin{vmatrix} dx & dy & 0 \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix} = \iint\limits_{S(L)} \frac{\partial \mathbf{B}}{\partial t}\,d\mathbf{S} - \oint\limits_{L} (\mathbf{v}\times\mathbf{B})\,d\mathbf{L}$$

and this proves the correctness of formula (5) in the case of the plane region $S(L) = D(L)$.

3. *Rotary motion.* We consider the rotation of curve L around a straight line lying in the plane of curve L. In what follows by $0XYZ$ we denote the immobile system, and by $0X'\,Y'\,Z'$ the mobile system rigidly connected with curve L. It is moreover assumed that curve L lies in the plane $0X'\,Y'$ and that the rotation axis coincides with the $0X$ axis (Fig. 2).

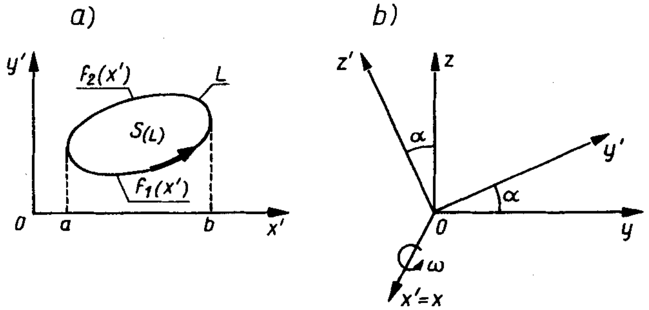

Fig. 2. a) — position of curve L on mobile plane $0X'\,Y'$; b) — rotation of mobile system $0X'\,Y'\,Z'$ around axis $0X$

α — rotation angle, ω — angular velocity, $S(L)$ — plane region spread on curve L; arrow indicates orientation of curve L

From Fig. 2 there result formulas defining the position of an arbitrary point of the mobile plane $0X'Y'$ in the immobile system $0XYZ$

$$x = x'; \quad y = y' \cos\alpha; \quad z = y' \sin\alpha. \tag{28}$$

Hence, after differentiation over time we get the coordinates of the linear velocity vector $\mathbf{v} = (v_x; v_y; v_z)$

$$\begin{aligned} v_x &= \frac{dx}{dt} = 0 \\ v_y &= \frac{dy}{dt} = -y' \sin\alpha \cdot \frac{d\alpha}{dt} = -\omega z; \qquad \omega = \frac{d\alpha}{dt} \\ v_z &= \frac{dz}{dt} = y' \cos\alpha \cdot \frac{d\alpha}{dt} = \omega y; \qquad \omega = \frac{d\alpha}{dt}. \end{aligned} \tag{29}$$

It is worth noting that formulas (29) may be obtained from the dependence $\mathbf{V} = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega} = (\omega; 0; 0)$ and $\mathbf{r} = (x; y; z)$. Formulas (29) will be used to determine the derivative of magnetic flux $\Phi = \iint_{S(L)} \mathbf{B}d\mathbf{S}$. In this integral the orientation of curve L and of the vector $d\mathbf{S}$ coincides with that of the dextrorotary screw. It results from Fig. 2 that vector $d\mathbf{S}$ has the following coordinates in the immobile system $0XYZ$

$$d\mathbf{S} = (0; -dx\,dz; dx\,dy). \tag{30}$$

Since vector $\mathbf{B} = (B_x; B_y; B_z)$ is written in the immobile system $0XYZ$, flux Φ is of the form

$$\Phi = \iint_{S(L)} \mathbf{B}d\mathbf{S} = \iint_{D_{xz}} -B_y\,dx\,dz + \iint_{D_{xy}} B_z\,dx\,dy \tag{31}$$

where D_{xz} — projection of plane $S(L)$ onto surface $0XZ$, D_{xy} — projection of plane $S(L)$ onto surface $0XY$. From Fig. 2 it results that regions D_{xz} and D_{xy} may be described as follows

$$\begin{aligned} D_{xz} &= \{(x; z): a \leqslant x \leqslant b; f_1(x) \sin\alpha \leqslant z \leqslant f_2(x) \sin\alpha\} \\ D_{xy} &= \{(x; y): a \leqslant x \leqslant b; f_1(x) \cos\alpha \leqslant y \leqslant f_2(x) \cos\alpha\}. \end{aligned} \tag{32}$$

Hence in the system $0XYZ$, the flux $\Phi = \Phi(t)$ described by (31) is of the form

$$\Phi(t) = \int_a^b dx \int_{f_1(x)\sin\alpha}^{f_2(x)\sin\alpha} -B_y(x; y; z; t)\,dz + \int_a^b dx \int_{f_1(x)\cos\alpha}^{f_2(x)\cos\alpha} B_z(x; y; z; t)\,dy. \tag{33}$$

In order to determine the derivative $\dfrac{d\Phi}{dt}$ we denote

$$(34)\qquad \begin{aligned} \Phi_1 &= \int\limits_a^b dx \int\limits_{f_1(x)\sin\alpha}^{f_2(x)\sin\alpha} -B_y(x;y;z;t)\,dz; \qquad y = c\,\mathrm{ctg}\,\alpha \\ \Phi_2 &= \int\limits_a^b dx \int\limits_{f_1(x)\cos\alpha}^{f_2(x)\cos\alpha} B_z(x;y;z;t)\,dy; \qquad z = y\,\mathrm{tg}\,\alpha. \end{aligned}$$

The relation between coordinates y and z results from (28) and represents the equation of rotating plane $0X'Y'$ in the system $0XYZ$ (Fig. 2). After applying twice the theorem (7) to integral Φ_1 we obtain

$$(35)\qquad \frac{d\Phi_1}{dt} = -\int\limits_a^b dx\,\Big\{B_y(x;f_2(x)\cos\alpha;f_2(x)\sin\alpha;t)\cdot f_2(x)\cos\alpha\cdot\frac{d\alpha}{dt}+$$

$$-B_y(x;f_1(x)\cos\alpha;f_1(x)\sin\alpha;t)\cdot f_1(x)\cos\alpha\cdot\frac{d\alpha}{dt}+$$

$$+\int\limits_{f_1(x)\sin\alpha}^{f_2(x)\sin\alpha}\left[\frac{\partial B_y}{\partial t}+\frac{\partial B_y}{\partial y}\cdot\frac{(-z)\,\dfrac{d\alpha}{dt}}{\sin^2\alpha}\right]dz\Big\}.$$

Hence, in view of (28), (29) and (32), the dependence (35) takes the brief form

$$(36)\qquad \frac{d\Phi_1}{dt} = \oint\limits_L B_y v_z\,dx + \iint\limits_{D_{xz}} -\frac{\partial B_y}{\partial t}\,dx\,dz + \iint\limits_{D_{xz}} \frac{\partial B_y}{\partial y}\cdot\frac{(-v_y)}{\sin^2\alpha}\,dx\,dz.$$

Proceeding analogously in determining the derivative $\dfrac{d\Phi_2}{dt}$ we get

$$(37)\qquad \frac{d\Phi_2}{dt} = \int\limits_a^b dx\,\Big\{B_z(x;f_2(x)\cos\alpha;f_2(x)\sin\alpha;t)\cdot f_2(x)(-\sin\alpha)\cdot\frac{d\alpha}{dt}+$$

$$-B_z(x;f_1(x)\cos\alpha;f_1(x)\sin\alpha;t)\cdot f_1(x)(-\sin\alpha)\cdot\frac{d\alpha}{dt}+$$

$$+\int\limits_{f_1(x)\cos\alpha}^{f_2(x)\cos\alpha}\left[\frac{\partial B_z}{\partial t}+\frac{\partial B_z}{\partial z}\cdot\frac{y\cdot\dfrac{d\alpha}{dt}}{\cos^2\alpha}\right]dy$$

i.e.

$$\frac{d\Phi_2}{dt} = \oint_L (-B_z v_y)\, dx + \iint_{D_{xy}} \frac{\partial B_z}{\partial t}\, dx\, dy + \iint_{D_{xy}} \frac{\partial B_z}{\partial z} \cdot \frac{v_z}{\cos^2 \alpha}\, dx\, dy. \tag{38}$$

After adding by sides (36) and (38) we get the equation of the derivative of the flux of vector **B** through the plane surface $S(L)$

$$\frac{d\Phi}{dt} = \oint_L (B_y v_z - B_z v_y)\, dx + \iint_{D_{xz}} -\frac{\partial B_y}{\partial t}\, dx\, dz + \iint_{D_{xy}} \frac{\partial B_z}{\partial t}\, dx\, dy + \tag{39}$$

$$+ \iint_{D_{xz}} \frac{\partial B_y}{\partial y} \cdot \frac{(-v_y)}{\sin^2 \alpha}\, dx\, dz + \iint_{D_{xy}} \frac{\partial B_z}{\partial z} \cdot \frac{v_z}{\cos^2 \alpha}\, dx\, dy.$$

Further transformations will be confined to the last two double integrals of the r.h.s. of (39). It results from (28) that $z = y \tan \alpha$, and so the Jakobian of transformation $x = x$ and $z = y \tan \alpha$ takes the value $J(x, y) = \tan \alpha$. This serves as a basis for the transformation of the next double integral

$$\iint_{D_{xz}} \frac{\partial B_y}{\partial y} \cdot \frac{(-v_y)}{\sin^2 \alpha}\, dx\, dz = \iint_{D_{xy}} \frac{\partial B_y}{\partial y} \cdot \frac{(-v_y)}{\sin \alpha \cos \alpha}\, dx\, dy. \tag{40}$$

From (29) we then have

$$\frac{-v_y}{\sin \alpha} = \frac{v_z}{\cos \alpha} = \omega y' = v; \qquad v = |\mathbf{v}| \tag{41}$$

and hence, on the basis of (40), we may write the equation

$$\iint_{D_{xz}} \frac{\partial B_y}{\partial y} \cdot \frac{(-v_y)}{\sin^2 \alpha}\, dx\, dz + \iint_{D_{xy}} \frac{\partial B_z}{\partial z} \cdot \frac{v_z}{\cos^2 \alpha}\, dx\, dy = \tag{42}$$

$$= \iint_{D_{xy}} \frac{v}{\cos \alpha} \left(\frac{\partial B_z}{\partial z} + \frac{\partial B_y}{\partial y} \right) dx\, dy = -\frac{1}{\cos \alpha} \iint_{D_{xy}} v\, \frac{\partial B_x}{\partial x}\, dx\, dy.$$

In the last transformation use was made of condition (6). Thus, after taking into consideration (30) and (42) and rearranging terms, the r.h.s. or (39) takes the form

$$\frac{d\Phi}{dt} = \iint_{S(L)} \frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{S} + \oint_L (B_y v_z - B_z v_y)\, dx - \frac{1}{\cos \alpha} \iint_{D_{xy}} v\, \frac{\partial B_x}{\partial x}\, dx\, dy. \tag{43}$$

The further considerations will concern the change of the double integral over region D_{xy} into a curvilinear integral. In agreement with the assumptions, the plane region $S(L)$ (Fig. 2a) is normal to the axes $0X$ and $0Y$. It results from this that the projection of region $S(L)$ onto the plane $0XY$ is normal to the axes $0X$ and $0Y$, i.e.

$$D_{xy} = \{(x; y): c \leqslant y \leqslant d;\ x_1(y) \leqslant x \leqslant x_2(y)\}. \tag{44}$$

Hence, after taking into consideration (28), (41) and (44) we get

$$\frac{1}{\cos\alpha} \iint_{D_{xy}} v \frac{\partial B_x}{\partial x} dx\,dy = \int_c^d \frac{\omega y'}{\cos\alpha} dy \int_{x_1(y)}^{x_2(y)} \frac{\partial B_x}{\partial x} dx = \int_c^d \frac{v}{\cos\alpha} \big[B_x(x_2(y); y; z; t) - B_x(x_1(y); y; z; t)\big] dy; \qquad z = y \operatorname{tg}\alpha. \tag{45}$$

Hence

$$\frac{1}{\cos\alpha} \iint_{D_{xy}} v \frac{\partial B_x}{\partial x} dx\,dy = \oint_L \frac{vB_x}{\cos\alpha} dy. \tag{46}$$

Since

$$\frac{1}{\cos\alpha} = \cos\alpha + \sin\alpha \tan\alpha$$

we get, after taking into consideration (41) and $z = y\tan\alpha$,

$$\oint_L \frac{vB_x}{\cos\alpha} dy = \oint_L (v\cos\alpha B_x + v\sin\alpha B_x \cdot \operatorname{tg}\alpha)\,dy = \oint_L B_x v_z\,dy + (-B_x v_y)\,dz. \tag{47}$$

Thus, in view of (46), the r.h.s. of (43) takes the form

$$\frac{d\Phi}{dt} = \iint_{S(L)} \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S} + \oint_L (B_y v_z - B_z v_y)\,dx - B_x v_z\,dy + B_x v_y\,dz \tag{48}$$

i.e.

$$\frac{d\Phi}{dt} = \iint_{S(L)} \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S} - \oint_L \begin{vmatrix} dx & dy & dz \\ 0 & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix} = \iint_{S(L)} \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S} - \oint_L (\mathbf{v} \times \mathbf{B})\,d\mathbf{L}.$$

This proves the correctness of (5) in the case of the plane region $S(L)$.

It is worth noting that (5) is true for an arbitrary smooth region $S'(L)$ spread on curve L. From condition (6) it results that

$$\operatorname{div} \frac{\partial \mathbf{B}}{\partial t} = 0. \tag{50}$$

Hence from (6) and the Gauss theorem we have

$$\Phi(t) = \iint_{S(L)} \mathbf{B} d\mathbf{S} = \iint_{S'(L)} \mathbf{B} d\mathbf{S}; \quad \iint_{S(L)} \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S} = \iint_{S'(L)} \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S}. \tag{51}$$

Formula (5) is thus correct for an arbitrary smooth surface $S'(L)$ spread on curve L.

3. Conclusions. We have considered the computationally simplest cases of closed conductor motion, namely translatory motion and rotation around a straight line lying in the conductor plane. It was found that in the case of such motion formula (5) is usually valid when condition (6) is satisfied. We did not consider cases like rotation around a straight line outside the conductor plane, spherical motion, general motion or conductor deformation. These problems are more complex and will be the subject of future works.

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Б. Сьвистач, **Нерелятивное изменение магнитного потока, охваченного замкнутым приводом. Прогрессивное и ротационное движение**

В настоящей статье доказывается уравнение

$$\frac{d}{dt}\iint\limits_{S(L)}\mathbf{B}d\mathbf{S}=\iint\limits_{S(L)}\frac{\partial\mathbf{B}}{\partial t}d\mathbf{S}-\oint\limits_{L}(\mathbf{v}\times\mathbf{B})\,d\mathbf{L}$$

в случае неподвижного контура L, который совершает прогрессивное и ротационное движение. Ось вращения расположена на плоскости контура L.