SELECTED PROBLEMS OF CIRCUIT THEORY 1

based on "*Circuit Analysis and Synthesis II*" Prof. C. Baker, Dr. Y. Yang Dr. A. Fernandez

SYNTHESIS OF CIRCUITS

Course contents:

1) Introduction
2) Passive circuits I - Realization of 1-ports.
3) Passive circuits II - Realization of 2-ports.
4) Active filters.
5) Digital filters.

Detailed content:

Passive Circuits

- Strategy of filter design
- System functions, realizability. Hurwitz polynomials
- Passive One-port networks
  - Cauer synthesis procedure
- Passive Two-port networks
  - Prototype filters
  - Impedance scaling
  - Frequency scaling
  - Transient response
  - Two-port reactive circuit synthesis
  - Butterworth and Chebyshev approximations
  - Frequency transformations

Active circuits

- Operational amplifiers
- Active two-terminal networks
  - Impedance converters and inverters, gyrator, negative elements
- Active filters
  - 2nd order sections. Sallen-Key circuit, Universal Active-RC Biquad

Digital filters

- Recursive and non-recursive filters. Aliasing. Windows.
- Digital transfer function, Z-transform
- Design of non-recursive filters
- Design of recursive filters

Suggested Bibliography

A. Papoulis "Circuits and Systems - A Modern Approach", Polytechnic Institute of New York, 1980.

H. Baher: Synthesis of Electrical Networks, New York: J. Wiley, 1984.

W-K. Chen "Passive and Active Filters – Theory and implementations", Wiley, 1986.

M.E. van Valkenburg "Analog Filter Design". HRW, 1982. (Good for the active filters part of the course).

C.S. Williams "Designing Digital Filters", Prentice Hall, 1986.

T. Kurowski, S. Taranow , "Selected Issues in the Theory of Linear and Non-Linear Systems", University of Zielona Gora, 2004 (English-Polish). **318493**.

additionally, the following books can be useful as reference material:

F.F. Kuo "Network Analysis and Synthesis". Wiley, 1966. (Useful for Passive network synthesis).

P. Bowron and F.W. Stephenson. "Active filters for communications and instrumentation", McGraw Hill

R.W. Hamming "Digital Filters". Prentice Hall, 1989. (Useful as introduction to digital filters).

G.G. Temes and W. LaPatra. "Circuit synthesis and design", McGraw Hill, 1977.

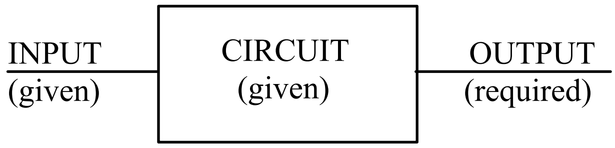
A. Budak. "Passive and active network analysis and synthesis", Houghton Mifflin, 1974.

R. Bracewell. "The Fourier transform and its applications", McGraw Hill, 1978.

INTRODUCTION

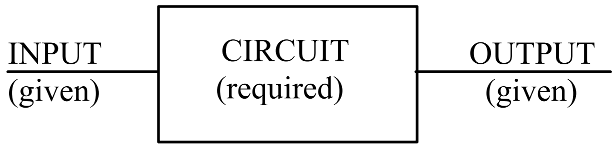
What is synthesis?

Let's start with analysis which seems simpler to understand:
A typical "analysis problem" is represented as:



- There is always one answer (existence theorem)
- Invariably this answer is unique. (uniqueness theorem)

Now with synthesis the situation is far less clear. In this case we have:



We now have that:

- Sometimes there is no answer and
- Often the answer is not unique.

In analysis there are standard tools for a systematic approach:

- circuit theory
- Fourier/Laplace transforms
- convolution.

In synthesis there are far too many design approaches to enumerate here, and plenty of scope for anyone to invent new ones. It is a less clearly defined problem and then, there are many ways to approach a solution. There is need of judgement as well as calculation. It is (arguably at least) more difficult, more open-ended, more interesting than analysis. Yet analysis remain vital - to test the design ideas before putting them into practice.

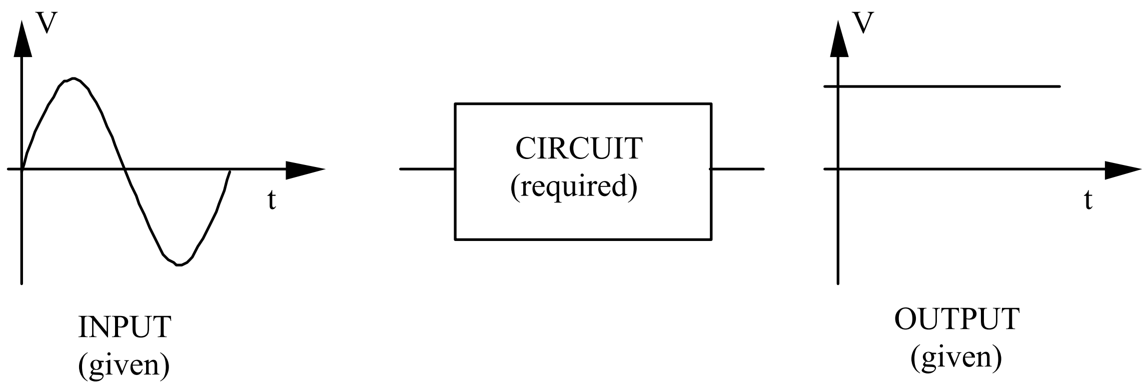
Because so much of synthesis is concerned with filter design, it is easy to suppose that all design is restricted to the frequency domain. Not so. Often there are clear requirements in both domains, and usually they conflict. For example, a filter's performance may be specified as a particular pass-band in the frequency domain, and at the same time it may be required to have a restricted transient response. This could occur in a radar system where bursts of oscillation are used, and the response to one burst must die away before the next pulse is received.

Usually the requirements are in conflict and a synthesis procedure consists in finding a compromise. In the last example, the better the discrimination is made in the frequency domain, the worse is the transient response.

Examples of the need for synthesis

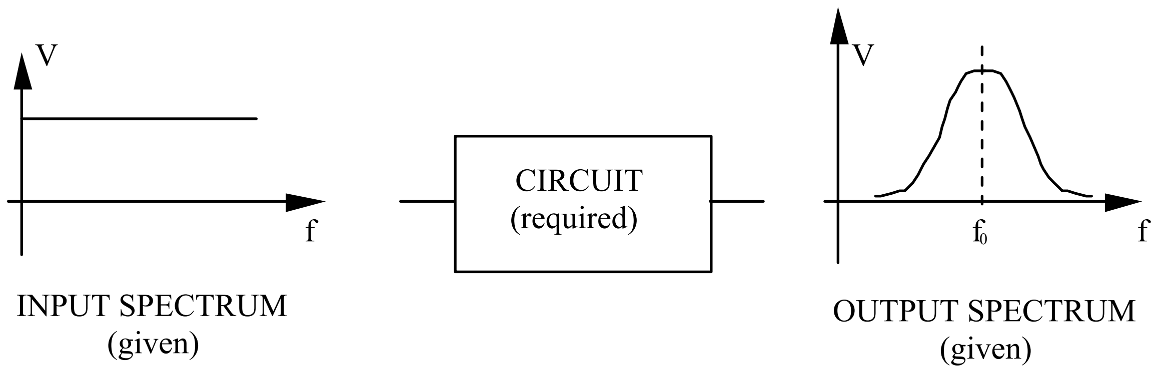
(1) A smoothed DC power supply:

The problem is by no means a trivial one. The response of the circuit to a sine wave is required to be a constant output.

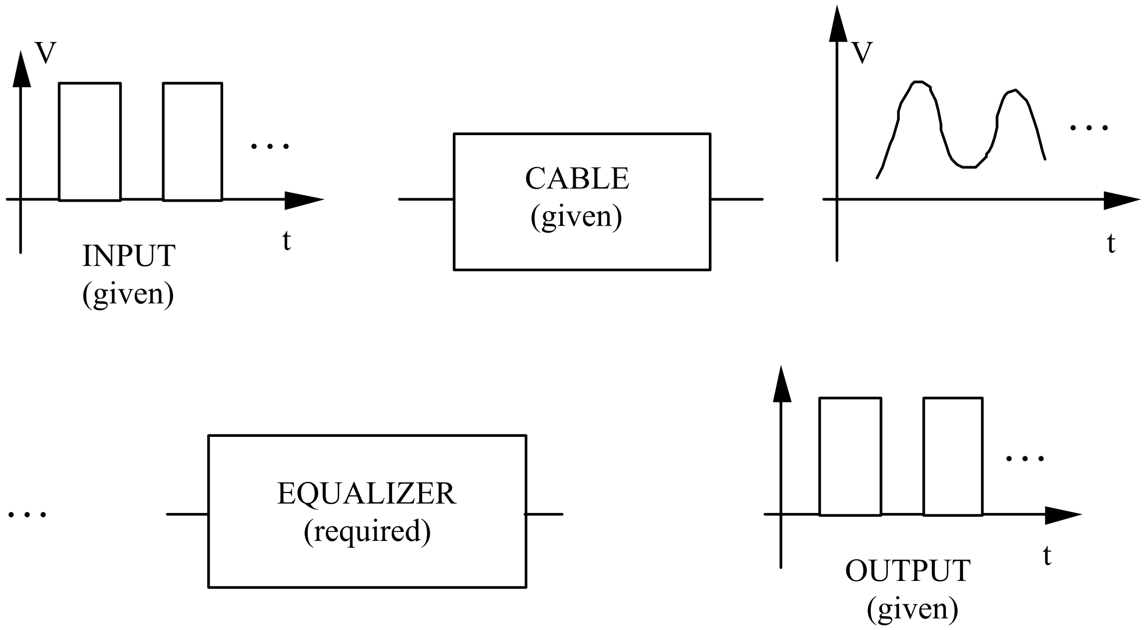


(2) A bandpass filter:

In this case the circuit only lets through frequency components within a certain frequency range.



(3) An equalizer (which is a circuit used to compensate for an undesired distortion or loss, *e.g.* to compensate for cable attenuation in a transmission system):



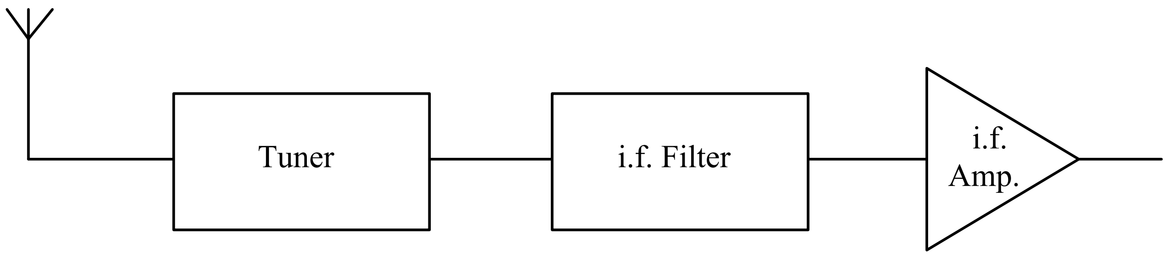
The diagram is trying to convey the idea that a pulse train should ideally be transmitted through the cable undistorted, but the cable characteristics are such that different frequency components are transmitted with different velocities so that they arrive at different times giving a distorted waveform (dispersion). The equalizer is a circuit whose characteristics are the inverse of the cable characteristics - leading, we hope, to a faithful pulse train at the receiving end.

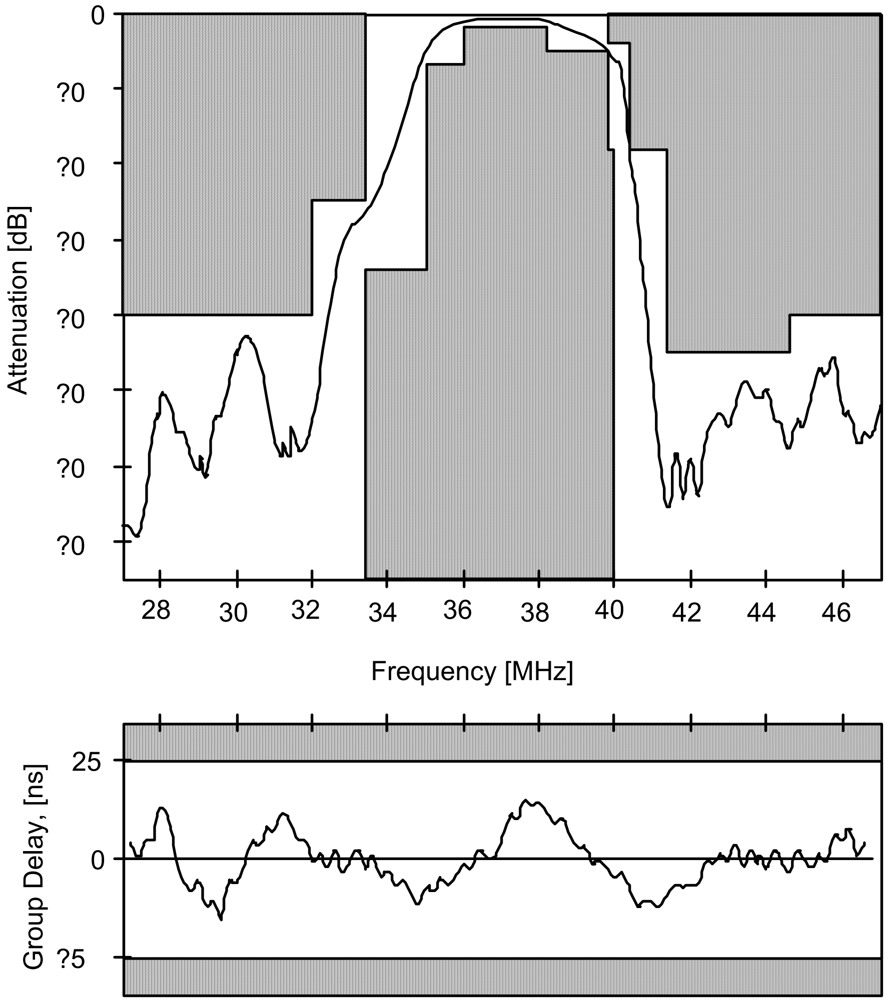
(4) Analysis of sampled data:

This is at first sight a less obvious candidate for synthesis procedures. Yet the concept of a filter is very general and very powerful. When we calculate the average of a set of readings (measurements), we are "smoothing out the fast variations" (high frequency components) — we are finding the "DC component" and this process is a numerical low-pass filter. The analogy with frequency-domain filters becomes closer when we envisage working out, say, the average of the last five temperature readings in a series of continuously sampled measurements. Or for instance when the annual inflation rate is updated every month. The general theme of digital filtering encompasses any linear operation on the data, and includes integration and differentiation.

(5) I.F. filter for a TV receiver

In this case, as in most practical cases, the output is not fixed specifically. Instead we only fix some restrictions to its shape. In all practical cases we are not interested in obtaining a predetermined fixed function as filter characteristic. Instead, we want an approximation to an ideal case that satisfy certain restrictions. This is particularly suitable since as the problem of synthesis not always has an exact solution, this has to be found as an approximation within a set of "realizable functions".

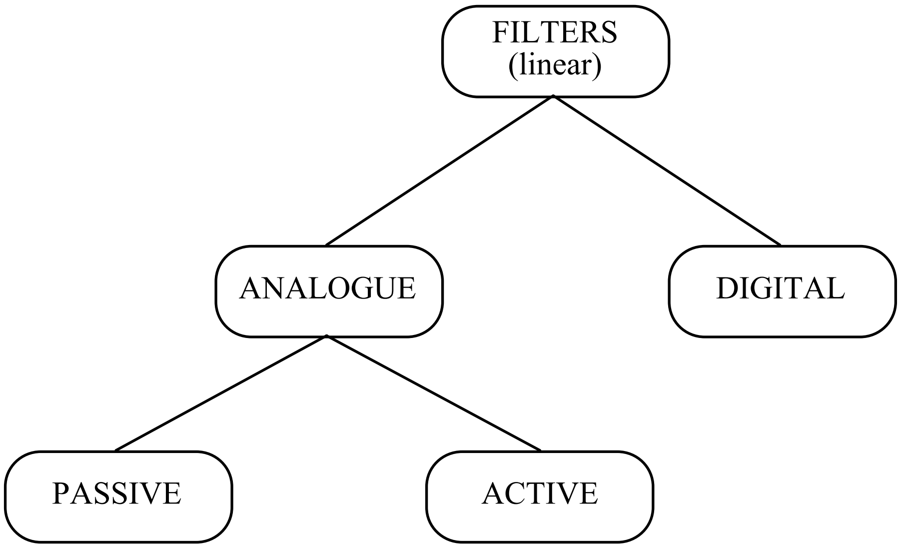




Tolerance masks for the amplitude response and for the group-delay of a television i.f. filter. The figure also shows a typical response (amplitude and phase) satisfying the requirements.

The available elements for the synthesis

These are:
- Passive circuits,
- active circuits and
- digital filters.



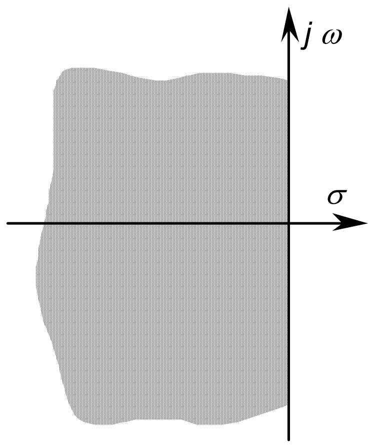
Passive circuits:

Components: These are restricted to the use of normal L,C and R electrical components (lumped or distributed) for the use in filters, equalizers, etc.

Causality: The response of these networks is determined by the signal information reaching them up to the output instant, *i.e.*, they are limited by causality: the effect cannot precede the cause.

Limited memory: Another restriction is that of severely limited memory: The energy dissipation time constants place restrictions on how much data from the past can influence the output at $t = 0$, *i.e.* the present.

Stability: Another restriction (which is also sometimes an advantage of passive circuits) is stability; *i.e.* they do not have energy sources and consequently there cannot be growing oscillations. This restriction means that poles and zeros of immittances (impedances or admittances) must be in the left-hand half-plane.



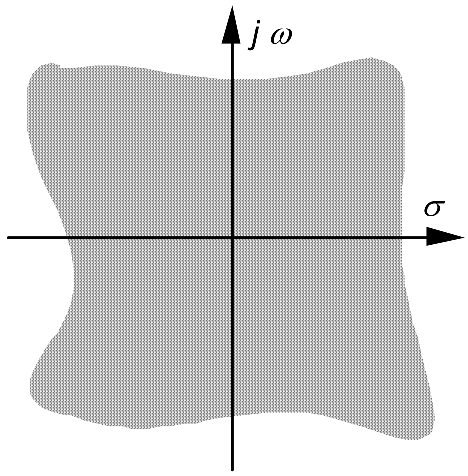
On the positive side, passive circuits are simple, reliable, stable, and they can handle high powers compared with circuits with op-amps.

On the negative side, inductors, which are often necessary, are not compatible with integrated circuit technology, where circuits are made more or less in a plane. Inductors are bulky and 3-dimensional. Another fundamental limit arises because, if you scale an inductor down in size, the Q-factor falls; a 10:1 linear scaling gives a 100:1 reduction in the Q-factor.

Active Circuits

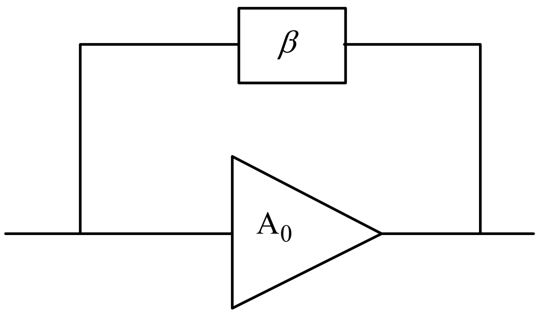
Components: Active circuits include an energy source, such as an amplifier, as well as resistors and capacitors. Inductors can and usually are avoided.

Active circuits are not limited by stability, – *e.g.* they can have an exponentially growing response (within certain limits). This means that poles and zeros of immittance functions can be located anywhere in the complex s-plane.

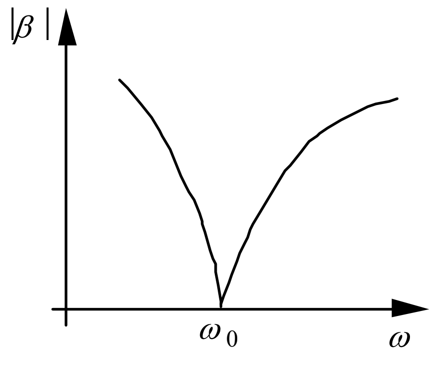


They can be made without the need of using inductors. This is particularly useful since in this way they are compatible with I.C. technology. Capacitors can be scaled down without affecting the Q-factor.

For example, a narrow-band filter, which would need inductors as a passive circuit, can be made as:

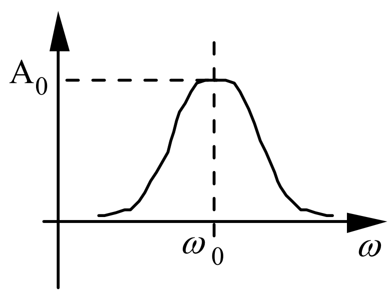


where



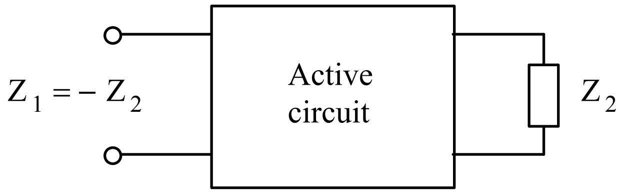
This leads to a frequency response: $\dfrac{V_2}{V_1} = \dfrac{A_0}{1-\beta A_0}$

where β is for instance, a twin-T network using only resistors and capacitors.

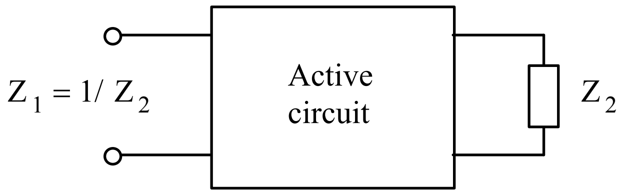


$Q \approx A_0/4$ for this circuit.

Active filters can be made to behave as if they contained negative resistance, capacitance and inductance. A negative resistance can be used to "cancel" losses and in the same way, negative capacitance can be used to "cancel out" unwanted capacitance, – and this is better than cancelling by the use of an inductor (tuning for resonance) because the latter would only work at one frequency. Negative R, L, and C can be simulated with a circuit:



Another useful device is an active circuit which "looks like an inductor" if the other port is terminated with a capacitor. It can effectively simulate inductors with capacitors, – very convenient for integrated circuits.



(gyrator)

Digital Filters

Digital filters are algorithms for digital computers or circuits. Digital filtering is any linear operation performed on data which has been sampled at equally spaced intervals.

It includes smoothing (averaging), integrating, separating signals (filtering) and predicting.

Examples are the Fourier transform (or Fast Fourier Transform), important in signal processing work, the Simpson rule or the trapezoidal rule of integration, the central difference formula of numerical derivative, *etc.* All these can be regarded as digital filters.

We can also find equivalent digital filters to analogue filters. Low-pass and other analogue filters have their digital counterparts. But digital filters have additionally some special properties which make them well suited for digital communication systems, especially when large distances are involved. It is then that analogue systems are at a particular disadvantage because the attenuation continuously degrades the signal-to-noise ratio. In digital systems, the signal is completely regenerated at intervals and then re-transmitted with no loss of information.

There are no impedance-matching problems in the digital domain. Also, two or more digital filters can have genuinely identical characteristics. These filters are also programmable so that their characteristics can be changed easily and rapidly - even almost continuously if needed.

Digital filters can have long memories if required. Initial conditions, far from dying away, can be stored indefinitely if needed. and the accuracy can be arbitrarily large, limited only by the word length and rounding/truncation error in the computer.

In some applications they can also “see the future”. For many data-processing applications, the information is all available on magnetic tape (or any other form of computer storage) before the calculations begin. In that sense, the filter algorithm can be written so as to take data from after as well as before the sampling interval for which the output is being calculated. Thus they are not limited by causality or stability.

SYNTHESIS OF PASSIVE CIRCUITS

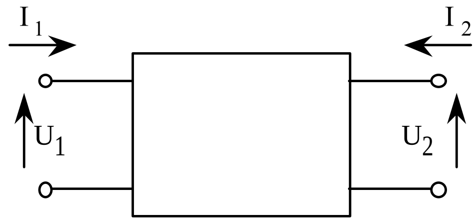
Introduction:

The only types of elements allowed are *R, L* and *C* (lumped or distributed). The constraints of stability and causality lead to mathematical conditions on the form of *Z(s)* and *Y(s)*.

Immittance is the word used generically for impedances and admittances.
In general, an immittance can be a driving point immittance or a transfer immittance.

U_1/I_1 : a driving point impedance

U_2/I_1 : a transfer impedance



Basic Ideas on Synthesis

Given the excitation *E(s)* to a system and the response *R(s)*, the desired system function *H(s)* is:

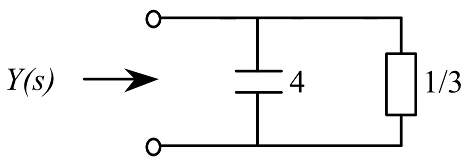
$$H(s) = \frac{R(s)}{E(s)}$$

which may be a driving point immittance, a transfer immittance or a voltage or current ratio.

We now have to:

- See whether this function is realizable as a passive network,
- if so, find the network or
- find an approximation.

For example, we might have $H(s) = I(s)/U(s) = 3 + 4s$ in which case the realization is simple:

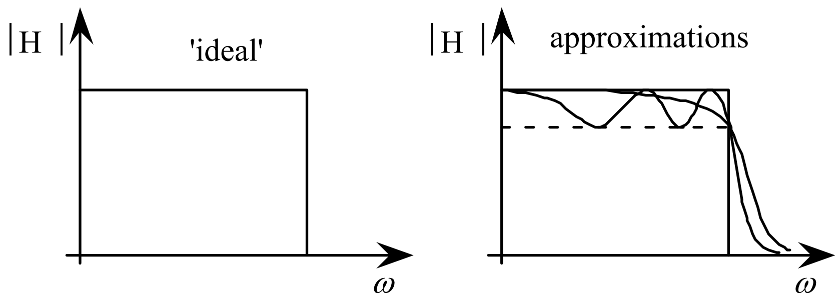


Synthesis of 2-ports:

The corresponding synthesis problem for a 2-port is transfer function design (filter design). A transfer function is a system function for which the variables are defined at different ports. *e.g.* if $I_1(s)$ is the excitation (input) and $U_2(s)$ is the response (output), the system function to design is the transfer impedance:

$$Z_{21}(s) = U_2(s)/I_1(s)$$

The most important aspect of transfer function synthesis is filter design. For example, an ideal amplitude response for a low-pass filter is:



As this "ideal" response is not realizable, this leads to the need for finding the "best" approximation.

Synthesis of Driving Point Immittances: (one-port circuits)

Physical Realizability of a D.P.I.

A network with a finite number of elements (*R, L, C*) has a driving point immittance of the form:

$$Z(s) \quad (\text{or } Y(s)) \quad = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0} \tag{1}$$

where all coefficients a_i and b_i are real.

Also, due to the constraints imposed by causality and stability, *Z(s)* (or *Y(s)*) must be what is called a <u>*positive real*</u> (p.r.) function; *i.e.*:

$$\text{Re}\{\, Z(s)\} > 0 \qquad \text{if Re}\{\, s\} > 0 \tag{2}$$

$$\text{Re}\{\, Z(s)\} \geq 0 \qquad \text{if Re}\{\, s\} = 0 \tag{3}$$

These conditions ((1),(2) and (3)) form necessary and sufficient conditions for the realizability of *Z(s)* (or *Y(s)*) with *R, L* and *C* components only.

The above set of conditions is very concise and constitutes an elegant way of defining realizability, but unfortunately it is not very useful from a practical point of view. To test for a rational function to be p.r., requires to analyse its behaviour for all values of $s \geq 0$, which is not simple.

If we re-write (1) in factorized form:

$$Z(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)} \tag{4}$$

A more convenient set of conditions can be obtained analysing the general properties of p.r. functions. Some of these are:

I) The constant K must be real and positive.
II) $|n-m| \leq 1$ (Also the lowest powers of numerator and denominator may differ at most by 1).
III) Poles (and zeros) must be either real or appear in conjugate pairs.
IV) Poles (and zeros) must lie on the left-hand half-plane or on the imaginary axis.
V) Poles on the imaginary axis must be single (but in conjugate pairs) and must have positive, real residues.
VI) $\text{Re}\{Z(j\omega)\} \geq 0$

Additionally, the sum of two p.r. functions is also p.r. (not necessarily the difference), and if *f(s)* is p.r., so is $1/f(s)$. Perhaps the simplest property to verify is that if we have a p.r. function written in the form (1), then all coefficients should be real and positive. Also, if the numerator or denominator are not odd or even polynomials, no coefficient corresponding to intermediate powers can be missing.

There is some redundancy in the above set but again, conditions I, IV, V and VI constitute a set of necessary and sufficient conditions and this expanded set is much easier to verify.

Condition I can be easily verified. The testing of condition IV is a bit more complicated:

In this case, we have that for a rational function to have no poles (or zeros) in the right-hand half space, both denominator and numerator must be of a class of polynomials called ***'Hurwitz'***. A *'Hurwitz'* polynomial is defined by the conditions:

1) P(s) real when s is real.
2) The roots of *P(s)* are in the left-hand half-plane or on the imaginary axis.

There is a simple way to test whether a polynomial is *Hurwitz* or not.

If we write $P(s) = m(s) + n(s)$ where $m(s)$ is the even part of $p(s)$ and $n(s)$ is the odd part, then the continued fraction expansion of the rational function:

$$R(s) = \frac{m(s)}{n(s)} = q_1(s) + \cfrac{1}{q_2(s) + \cfrac{1}{\dots + \cfrac{1}{q_n(s)}}}$$

has only positive coefficients q_i.

(We will see more about continued fraction expansions later).

(Of course we can get the same result if we factorise the function to form (4) where the position of each pole and zero are evident).

To check condition V we need a partial fraction decomposition and then we can check the residues. Finally, condition VI is simply tested examining the form of $F(j\omega)$.

All this constitutes a rigorous test of realizability. However, the easiest way to assess realizability of a function is simply trying to realize it.

Examples

a) $Z(s) = \dfrac{as^2 + b}{s} = as + \dfrac{b}{s}$ clearly realizable as a combination of L and C.

zeros are $\pm j\sqrt{b}/\sqrt{a}$ and poles at 0 and ∞.

b) $Z(s) = s + 3$ This is OK, (series connection of R and L) one zero at -3 and a pole at infinity.

c) $Z(s) = s + j$ NO, condition III is not satisfied.

d) $Z(s) = s^2 + 1 = (s + j)(s - j)$ NO, condition II is not satisfied ($n = 2$, $m = 0$).

e) $Z(s) = \dfrac{s^2 + 1}{s^3}$ NO, condition V is not satisfied.

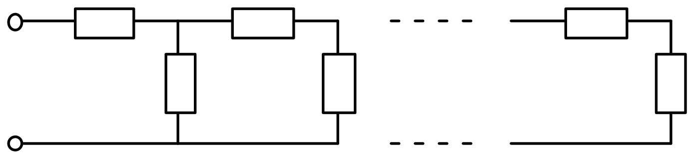
Methods of Synthesis of One-Port Networks

We will treat this case first because it is simpler than the synthesis of 2-ports. In addition, the synthesis of 2-ports can be usually reduced to the synthesis of a d.p.i. In fact, normally the specification for a 2-port circuit is in the form of a prescribed shape of V_{out}/V_{in} as a function of frequency and this leads to the network requiring a particular d.p. impedance or admittance in the form of a ratio of polynomials.

There are (at least) two standard approaches to the synthesis of d.p.i.: Foster and Cauer methods.

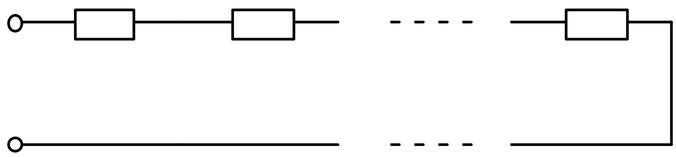
Cauer synthesis:

The procedure is based on the continued fraction expansion of the quotient of polynomials and leads to networks of the form:



Foster synthesis:

This is based in a partial fraction expansion and then, in the case of an impedance, the resultant network is a series connection of elements (or blocks formed with the parallel connection of two elements) as in:

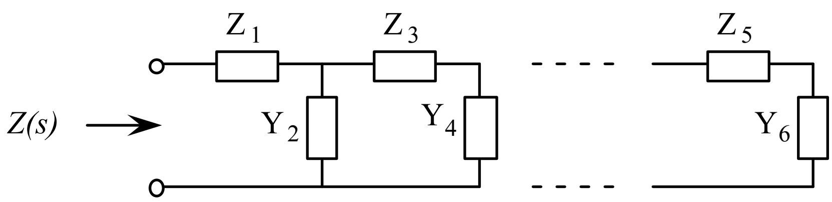


Similarly, in the case of admittances, the corresponding network is a parallel connection of single elements (or pairs of elements in series).

In the following we will only be concerned with the Cauer synthesis.

Cauer Synthesis

We start analysing a ladder network where each box represents either *R*, *sL* or *sC* or their corresponding reciprocals.



By inspection, we have:

$$Z(s) = Z_1 + \cfrac{1}{Y_2 + \cfrac{1}{Z_3 + \cfrac{1}{Y_4 + \cfrac{1}{Z_5 + \cfrac{1}{Y_6}}}}}$$

where the quotients are <u>alternately impedances or admittances</u>, moving from left to right into the ladder network. Since the impedance was specified, the first term must be an impedance, and the ladder starts with a series element. If the admittance were required in the form of a continued fraction, the first term would be an admittance and the ladder network would start with a shunt element.

> Obs. Every realizable d.p. immittance can be realized as a ladder network.

Now, to synthesize a ladder network from a given driving point immittance, we have to express *Z(s)* or *Y(s)* as a continued fraction. Each quotient represents a physically realizable component.

Example

Given the (realizable) impedance function: $$Z(s)=\frac{s^4+4s^2+3}{s^3+2s}$$

express it in the form of a continued fraction and so derive the ladder network, giving all component values.

The first point to note is that there can be more than one continued fraction (and consequently more than one ladder network) for the required *Z(s)*. Additionally, there may be some decompositions in the form of continued fraction which may not correspond to a physically realizable network.

a) We first take numerator and denominator in descending powers of s. We obtain the continued fraction expansion by repeated division and inversion:

$$Z(s)=\frac{s^4+4s^2+3}{s^3+2s}=Z_1+Z'_2=s+\frac{2s^2+3}{s^3+2s}$$ $Z_1 = s$ must be an impedance!

Now, $$Y'_2=\frac{1}{Z'_2}=\frac{s^3+2s}{2s^2+3}=Y_2+Y'_3=\frac{s}{2}+\frac{(1/2)\,s}{2s^2+3}$$ $Y_2 = s/2$

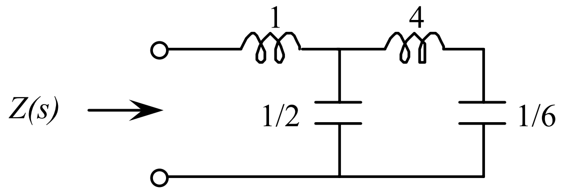
and again, $$Z'_3=\frac{1}{Y'_3}=\frac{2s^2+3}{(1/2)\,s}=Z_3+Z_4=4s+\frac{3}{(1/2)\,s}$$ $Z_3 = 4s$

and finally, $Y_4 = s/6$.

The complete expansion is then:

$$Z(s)=s+\cfrac{1}{(1/2)s+\cfrac{1}{4s+\cfrac{1}{(1/6)s}}}$$

and the corresponding network:



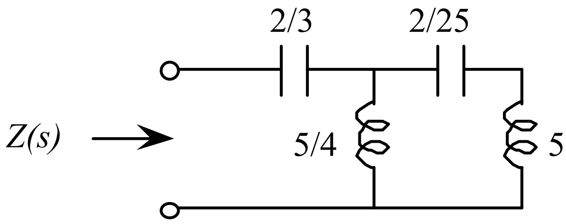
This process of successively extracting factors proportional to s is called "removing the poles at infinity".

b) An alternative synthesis comes from arranging the terms in ascending powers of s, or "removing the poles at the origin":

$$Z(s) = \frac{3 + 4s^2 + s^4}{2s + s^3} = \frac{3/2}{s} + \frac{(5/2)s^2 + s^4}{2s + s^3} \cdots, \qquad \text{etc.}$$

So the expansion and the corresponding circuit realization is:

$$Z(s) = \frac{3/2}{s} + \cfrac{1}{\cfrac{4/5}{s} + \cfrac{1}{\cfrac{25/2}{s} + \cfrac{1}{\cfrac{1/5}{s}}}}$$



The Euclid Algorithm

This is not more than a methodical style of book-keeping, that organizes the continued-fraction algebra.

In general, to find the continued fraction of a d.p.i. function there are always 4 ways of obtaining the expansion, of which 0, 1 or (at most) 2 can be successful:

If we have a d.p.i. in the form of an impedance as in:

Z(s) = *F(s)*, we can as before try in two ways, taking numerator and denominator in decreasing or increasing order of the powers of s. Also, we can invert the function taking instead:

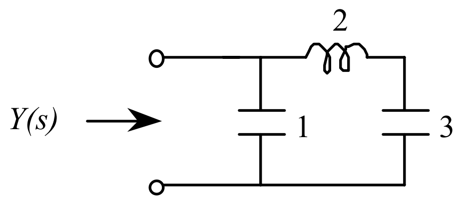
Y(s) = 1/*F(s)* (which of course should correspond to the same d.p. impedance as before) and try again in the two forms of ordering the powers of s.

If we consider the function $\quad Y(s) = \dfrac{6s^3 + 4s}{6s^2 + 1}$

the solution with the Euclid algorithm gives:

$$
\begin{array}{c|ccc|c}
\text{Y of shunt elements} & & \text{(a)} & & \text{Z of series elements} \\
s & 6s^3 + 4s & : & 6s^2 + 1 & \\
 & 6s^3 + s & & & \\
\cline{2-2}
 & 3s & \longrightarrow & 6s^2 & 2s \\
\cline{4-4}
3s & 3s & \longleftarrow & 1 & \\
\cline{2-2}
 & 0 & & &
\end{array}
$$

and the corresponding circuit is:



Now, reordering the powers of *s* in ascending order:

$$
\begin{array}{c|ccc|c}
\text{Y of shunt elements} & & \text{(b)} & & \text{Z of series elements} \\
4s & 4s + 6s^3 & : & 1 + 6s^2 & \\
 & 4s + 24s^3 & & & \\
\cline{2-2}
 & -\ 18s^3 & & &
\end{array}
$$

there is no solution since a negative residual is obtained.

If we try now inverting to realize the corresponding *Z(s)*:

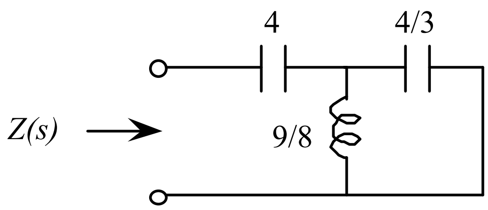
$$
\begin{array}{c|ccc|c}
\text{Y of shunt elements} & & \text{(c)} & & \text{Z of series elements} \\
 & 6s^2 + 1 & : & 6s^3 + 4s & 1/s \\
 & 6s^2 + 4 & & & \\
\cline{2-2}
 & -\ 3 & & &
\end{array}
$$

it does not work.

Finally, taking the ascending order in the inverted form:

Y of shunt elements	(d)		Z of series elements
	$1 + 6s^2$	$: \; 4s + 6s^3$	$1/(4s)$
	$1 + (6/4)s^2$		
$8/(9s)$	$(18/4)s^2$	$4s$	
	$(18/4)s^2$	$6s^3$	$3/(4s)$
	0		

And the corresponding circuit is:



Note that we have two possible realizations, both with identical response.

Exercise

Find ladder network(s) corresponding to: $Z(s) = \dfrac{6s^3 + 8s^2 + 4s + 4}{6s^2 + 8s + 1}$

All of the previous examples correspond to circuits containing only reactive elements. As we will see later, the realization of circuits containing resistance can be done following a similar procedure although it is a bit more complicated.

Driving Point Immittance of a Reactive (L, C) Network

If there are only L and C elements in the network, the d.p.i. has some special properties (or additional restrictions).

This kind of circuits (without R) is important in filter design where a transfer function without loss is usually desired.

For these circuits, the realizability conditions are more restrictive than for general RLC networks.

$L - C$ Realizability conditions:

1) $Z(s)$ or $Y(s)$ is the ratio of either an even to an odd polynomial or vice versa.

2) All poles and zeros alternate on the $j\omega$ axis.

3) All poles and zeros are simple and occur in conjugated pairs.

4) The residues at the poles $\lim\{(s - p_i)Z(s)\}$ must be real and positive.

5) The highest powers of s (and the lowest) in the numerator and denominator must differ exactly by 1.

6) There must be either a pole or a zero at the origin. (Obviously, the same can be said for infinity.)

There is visibly a lot of redundancy in the above constraints. A simpler and more concise statement comes from considering $Z(s)$ (or $Y(s)$) in a factorised form.

Algebraic Statement of *LC* Realizability

The d.p. immittance must be of one of the four types:

1) $$K\frac{s(s^2+a_1^2)(s^2+a_2^2)\cdots(s^2+a_n^2)}{(s^2+b_1^2)(s^2+b_2^2)\cdots(s^2+b_n^2)}$$ with $0 < b_1 < a_1 < \ \ldots\ < b_n < a_n$

zero at the origin, pole at infinity.

2) $$K\frac{s(s^2+a_1^2)(s^2+a_2^2)\cdots(s^2+a_n^2)}{(s^2+b_1^2)(s^2+b_2^2)\cdots(s^2+b_{n+1}^2)}$$ with $0 < b_1 < a_1 < \ \ldots\ < a_n < b_{n+1}$

zero at the origin, zero at infinity.

3) $$K\frac{(s^2+c_1^2)(s^2+c_2^2)\cdots(s^2+c_n^2)}{s(s^2+b_1^2)(s^2+b_2^2)\cdots(s^2+b_n^2)}$$ with $0 < c_1 < b_1 < \ \ldots\ < c_n < b_n$

pole at the origin, zero at infinity.

4) $$K\frac{(s^2+c_1^2)(s^2+c_2^2)\cdots(s^2+c_{n+1}^2)}{s(s^2+b_1^2)(s^2+b_2^2)\cdots(s^2+b_n^2)}$$ with $0 < c_1 < b_1 < \ \ldots\ < b_n < c_{n+1}$

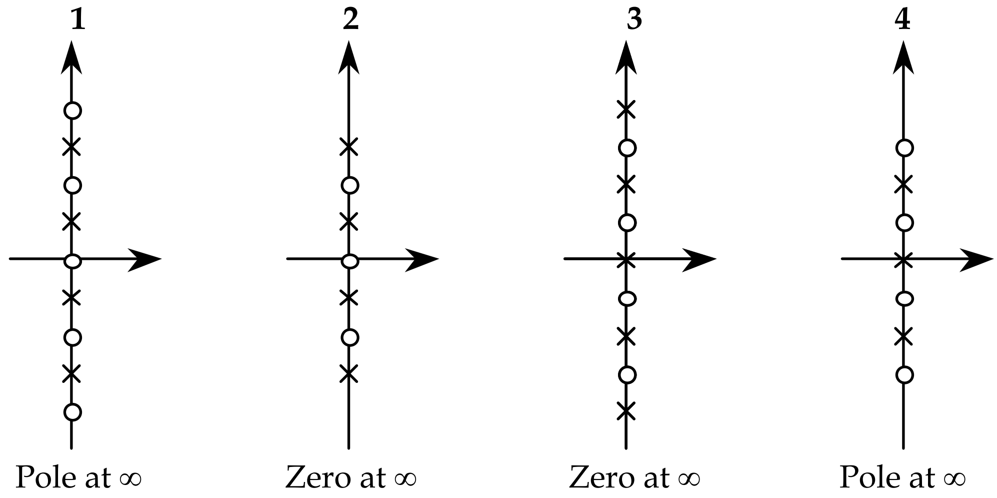
pole at the origin, pole at infinity.

Examples

$3\dfrac{(s^2+1)}{s(s^2+4)}$ is OK (type 3)

$7\dfrac{s(s^2+1)}{(s^2+4)}$ is not OK

The simplest way to display *LC* realizability is to say that, in the pole/zero and K representation of equation 4 in page 13; K is real and positive, and the pole/zero distribution along the $j\omega$ axis is one of the following 4 possibilities (same numbering as above):



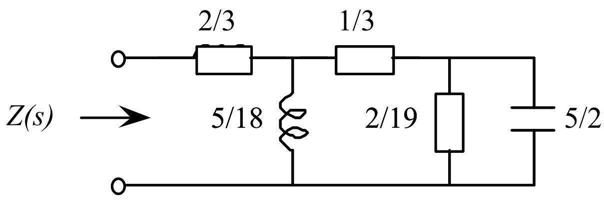
Networks containing Resistance

It is also possible to synthesize networks containing all three circuit elements using the Cauer method, *i.e.* obtaining a continued-fraction expansion leading to a ladder network, as for example in the following exercise.

Exercise

Use the Cauer method to derive the following circuit from:

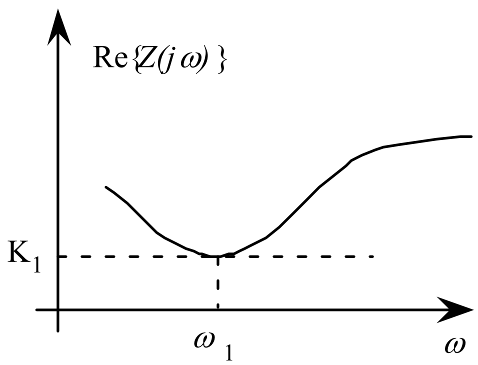
$$Z(s) = \frac{4 + 5s + s^2}{6 + 5s + s^2}$$



Important Note: In the synthesis of these cases, it is often necessary to rearrange the terms of the polynomials during the process in order to keep the quotients positive (*i.e.* realizable).

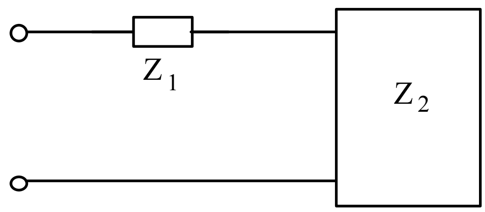
In the case of circuits containing resistances, another elementary synthesis procedure (apart from the removal of poles at the origin and at infinity) is the removal of min Re{ $Z(j\omega)$} .

If we have a function $Z(s)$ which has a minimum on the $j\omega$ axis at $\omega = \omega_1$:



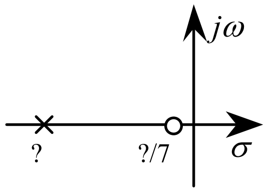
We can decompose Z(s) as:

$$Z(s) = Z_1(s) + Z_2(s)$$

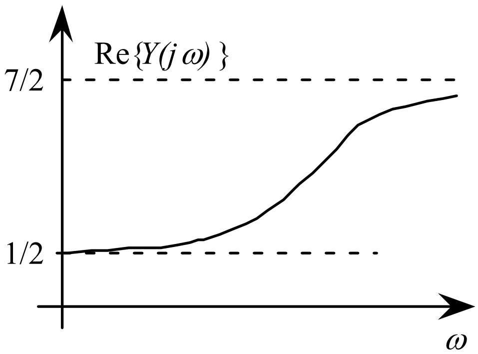


We have already seen the case where Z_1 corresponds to a pole at the origin (a/s) or at infinity (as). We can also extract a constant of value K provided that $K \le K_1$ since in that case the remainder will still satisfy the condition: $\mathrm{Re}\{Z_2(j\omega)\} \ge 0$.

Example



$Y(s) = \dfrac{7s+2}{2s+4}$ is a p.r. function:



We can see that:

$$\mathrm{Re}\{Y(j\omega)\} = \frac{4+7\omega^2}{8+2\omega^2}$$

and the minimum occurs at $\omega = 0$.

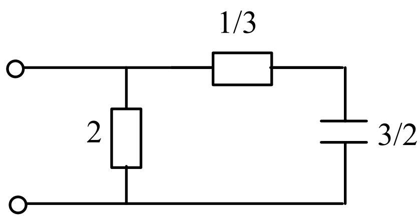
We can then remove $Y_1 = 1/2$ and the remainder is still p.r. (*e.g.* realizable)

$$Y(s) = 1/2 + Y_2\,; \qquad Y_2(s) = 3s\,/\,(s+2)$$

$Y_2(s)$ can be treated in the same way or, considering that $Y_2(0) = 0$, the pole of $Z_2(s) = 1/Y_2(s)$ at the origin can be removed. But, in this case it is simpler to recognise that:

$Z_2(s) = 1/3 + 2/3s$,

and then the complete network is:



Problem 1

Determine which of the following functions are realizable as driving point impedances. Find a ladder circuit realization whenever possible.

a) $\dfrac{s^2+s+4}{s+5}$ b) $\dfrac{s^5+20s^3+64s}{s^4+10s^2+9}$ c) $\dfrac{s^2+1}{(s+1)^2}$

d) $\dfrac{s^4+2s^3+3s^2+2s+1}{s^4+2s^3+2s^2}$ e) $\dfrac{s^2+7s+12}{s^2+3s+2}$

f) $\dfrac{s+4}{s^2+s+15}$ g) $\dfrac{s^6+6s^4+11s^2+6}{6s^5+24s^3+22s}$

SYNTHESIS OF PASSIVE TWO-PORTS NETWORKS (FILTER DESIGN)

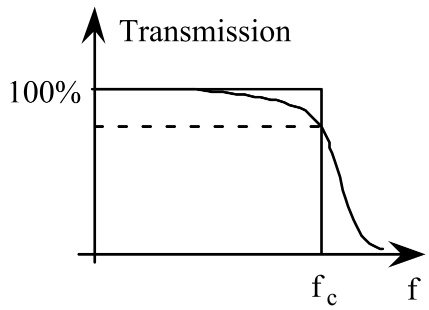
Transfer Function Synthesis

The synthesis of two-port networks can be easily reduced to the synthesis of a d.p.i. and then solved using the methods described earlier. However, one of the most important problems in filter design is that very rarely the specifications are in the form of a defined function of s as we have seen in the previous chapter. Instead, the specifications are usually given in terms of restrictions over the behaviour of the "frequency response" *i.e.* the shape of the curve (amplitude)^2 versus frequency or phase versus frequency. That is, some characteristics are given of a function of ω and to synthesize a network we will first need to find a precise function of s which behaves as prescribed on the $j\omega$ axis. Then, one of the important aspects of this process is the approximation.

Filter design strategy:

1.- From the specification to the transfer function

Often the specifications are given (or are transformed) in terms of a normalized low-pass prototype. This makes the synthesis procedure much easier and the resultant filter can be converted back to any other form of response and its element values re-scaled. In this terms, the specifications are given in the form of values to:



– cut-off frequency f_c

– maximum attenuation in the passband.

– rate of "fall-off" in the stop band.

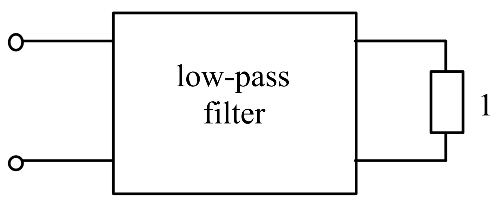
Scaling of component values

Impedance Scaling

If we start with a 'normalized' filter designed with easy numbers:

Impedance level = 1 Ohm

Cut-off frequency:
ω_c = 1 rad/sec



To change "impedance level" from 1 Ohm to say, R_0 Ohm, just multiply all individual impedances by R_0. Then the impedance scaling factor is $k_L = R_0$:

$R_{new} = k_L R_n$ where R_n, L_n and C_n are the values in the normalized filter.

$L_{new} = k_L L_n$

$C_{new} = C_n/k_L$

Frequency Scaling:

If we now want to change the frequency ω_c to ω_0 instead of 1 rad/sec:

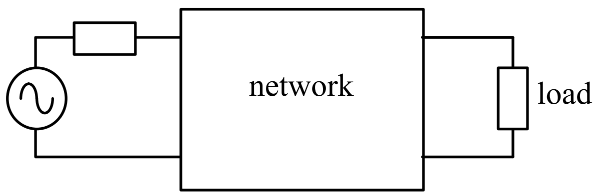
We want each component to have the same behaviour at $\omega_c = \omega_0$ as they had at 1 rad/sec, that is, we need the impedance values to be the same at the new frequency as they were for 1 rad/sec: So, the frequency scaling factor is $k_f = \omega_0$, and

R remains unchanged (since the impedance of resistors is independent of freq.)

$L_{new} = L_{norm}/k_f$ (to have: $\omega_0 L_{new} = 1\ L_{norm}$)

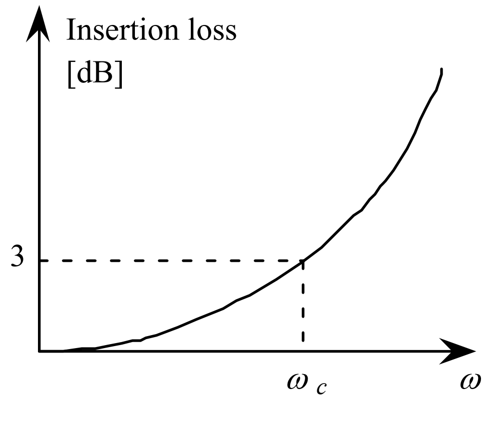
$C_{new} = C_{norm}/k_f$ (to have: $\omega_0 C_{new} = 1 C_{norm}$)

Another form of specifying a filter is using the so called "*Insertion Loss*" function:



The insertion loss (in dB) as a function of frequency is defined as:

$$\text{Insertion loss} = 10 \log_{10} \frac{\text{Power in load when network is absent}}{\text{Power in load when network is present}}$$



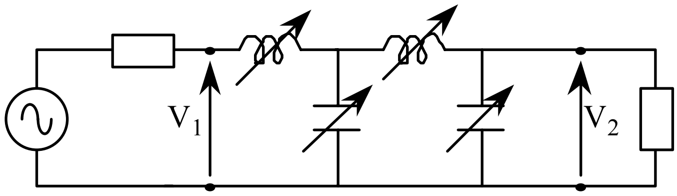
The cut-off frequency is the frequency at which some specified insertion loss is obtained *e.g.* 3 dB. The 3-dB point is the most common choice but it is not universal.

Again in this case the practical specifications are given in terms of restrictions to the behaviour of this function.

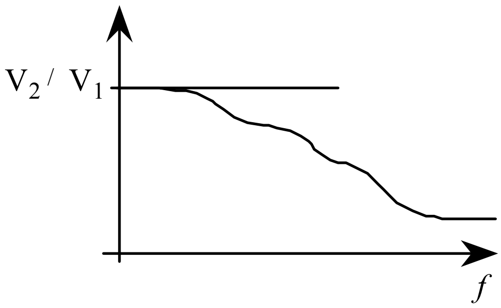
In order to synthesize a filter we have to use an approximation which often is a polynomial to fit the design curve. There are multiple choices where the most commonly used are the Butterworth, Chebyshev, or Bessel responses, each of them corresponds to a different choice of approximation polynomials. The ultimate choice should also consider other aspects of the filter performance as it is for example the transient response.

Choice of filter characteristics

Let's consider a low-pass filter with variable parameters:



The frequency response will change with the element values:



How can we adjust the element values to get the "best" characteristic?

What is the "best" response?

There are different common types of response or families of filters:

1) Maximally flat response (Butterworth filters)

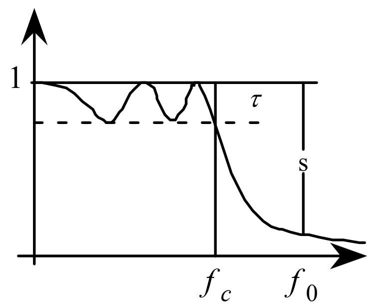
For a specific value of V_2 at a given ω_c, we ask for: $$\left.\frac{\partial^n V}{\partial \omega^n}\right|_{\omega=0} = 0$$

for n as large as possible.
The more derivatives equal to 0 at $\omega = 0$, the "flatter" the response results at the origin.

2) Equal ripple response (Chebyshev filters)

This can be characterized by the conditions or specifications:
For a given f_0, $V(f_0)$ and f_c: minimize the value of τ (the maximum ripple in the passband).
or equivalently:

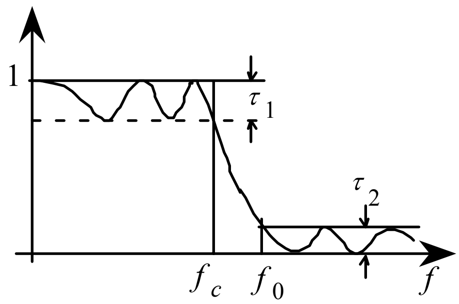
Given τ, f_0, f_c: minimize $V(f_0)$ (maximize the rate of fall-off).



The Chebyshev approximation will give the maximum rate of fall-off in the stop band for a maximum allowed ripple in the pass-band.

3) Elliptic filter

For given cutoff frequency f_c and ripple levels τ_1 and τ_1, minimize f_0.
or equivalently, (maximize the rate of fall-off between f_c and f_0.

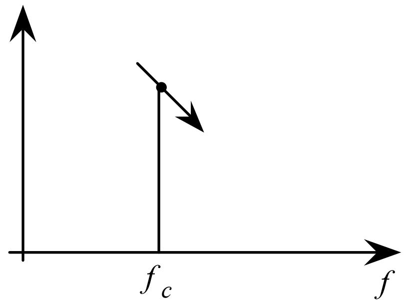


In this case we allow ripple in the stop-band as well as in the pass-band.

4) Optimum or "*L*" filter

The response is monotonic with ω

At a specified f_c, maximize: $\dfrac{\partial V}{\partial \omega}$



5) <u>Linear phase response</u> (Bessel filters)

In this case we want an approximation to a linear phase response. But now, why linear phase?

An ideal delay line (only introduces a time delay T) has a system function:

$$H(s) = K\, e^{-sT}$$

Then, the frequency response is: $H(j\omega) = K\, e^{-j\omega T}$; that is:

Amplitude response = K, a constant and

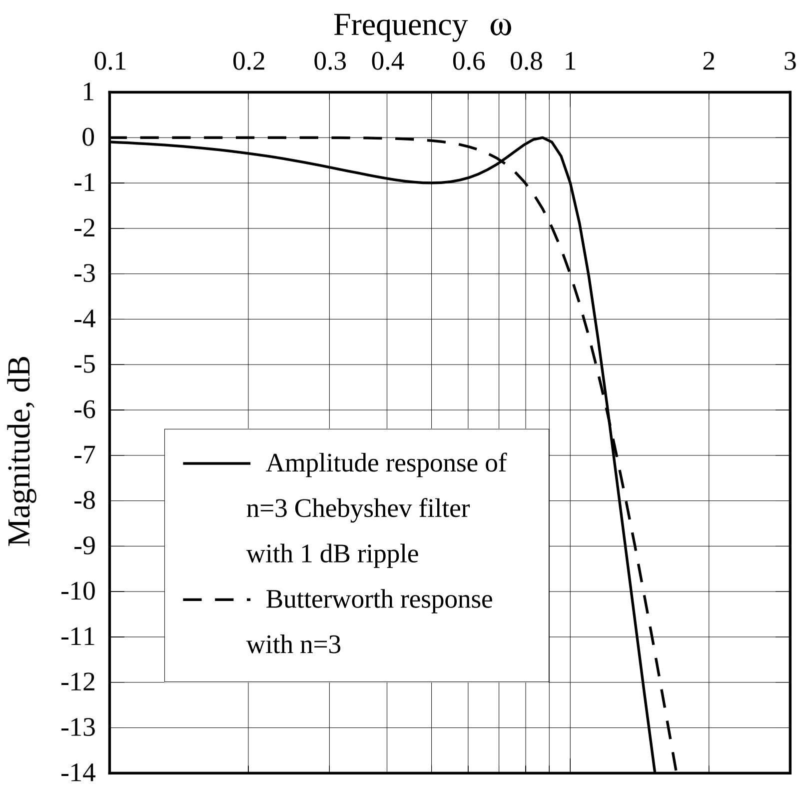
Phase response = $\phi(\omega) = -\omega T$ $\therefore$ varies linearly with ω.

If we excite with $e(t)$ (or $E(s)$), the response is:

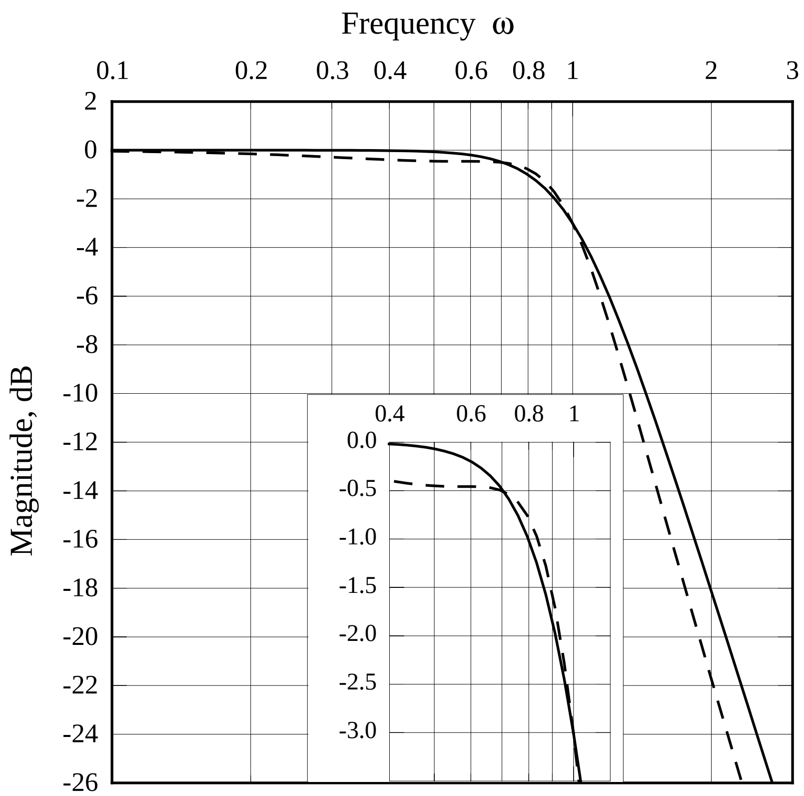
$$R(s) = K\, E(s)\, e^{-st} \qquad \text{and then,}$$
$$r(t) = K\, e(t-T)\, u(t-T),$$

which is the same function as in the input, but delayed by a time T.

Then a linear phase response will modify the amplitude but not the relative phase of each frequency component.



Amplitude responses of third order filters (n = 3).

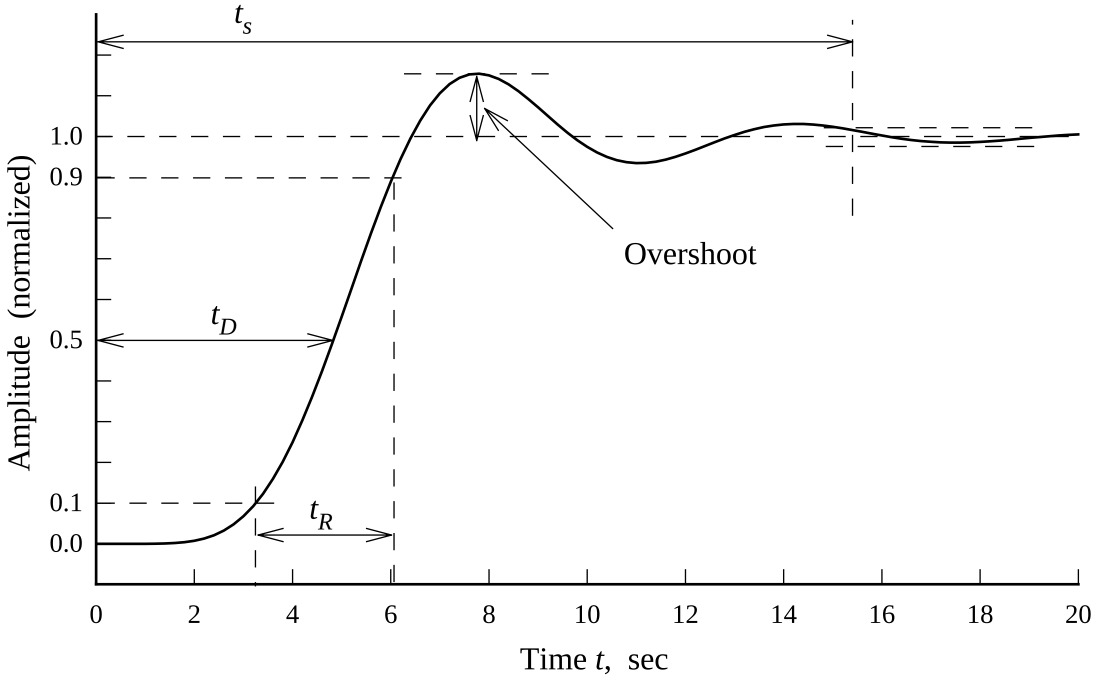


Amplitude responses of Butterworth filter of order 3 (solid line) and Optimum filter of order 3 (dashed line).

Transient Response of Low-Pass Filters

In this section we will compare the transient response of the filters discussed earlier. In particular, we will compare the step response of the filters according to a set of figures of merit. In relation to the general response to the step excitation, we can define the following parameters or figures of merit:

1. *Rise time* The rise time t_R of the step response is defined here as the time required for the step response to rise from 10% to 90% of the final value as shown in the figure.
2. *Ringing* Ringing is an oscillatory transient occurring in the response of the filter to a sudden change in input (such as a step). A measure of the ringing in a step response is given by its settling time.
3. *Settling time* This is the time t_s beyond which the step response does not differ from the final value in more than, say, ± 2%, as shown in the figure.
4. *Delay time, t_D* Delay time is the time which the step response requires to reach 50% of its final value.
5. *Overshoot.* The overshoot in the step response is the difference between the peak value and the final value, expressed as a percentage of the final value.



Figures of merit for step response.

Most of the filter characteristics mentioned earlier are related to the frequency response, in particular, to the bandwidth and to the phase linearity. The rise time and the delay time are closely related to each other but have little connection with overshoot.

We will now examine qualitatively these relationships.

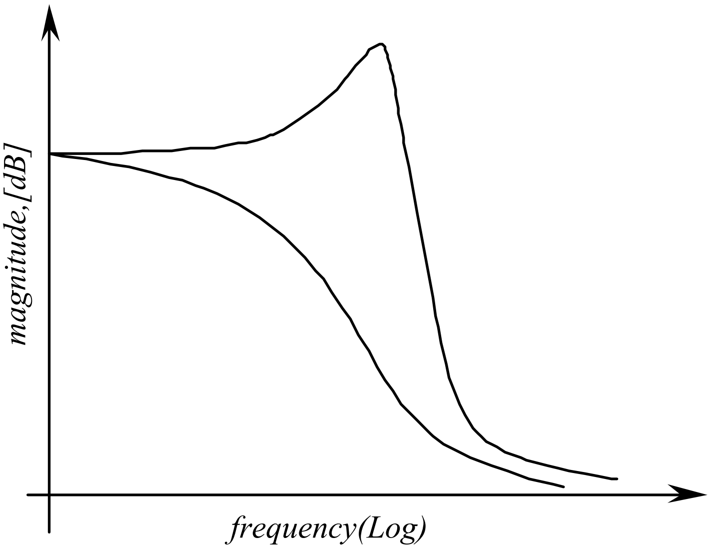
Rise time and bandwidth have an inverse relationship in a filter. The wider the bandwidth, the smaller is the rise time and vice versa. This could be understood by noting that a limited performance of a filter at high frequencies slows down the increase of the output in response to an abrupt step in the input, causing a long rise time.

It has been found experimentally that the relationship is very closely inversely proportional so that we can write the following expression:

$$T_R \times (\text{Bandwidth}) = \text{Constant}$$

This particular parameter, the rise time, is an important criterion in pulse transmission.

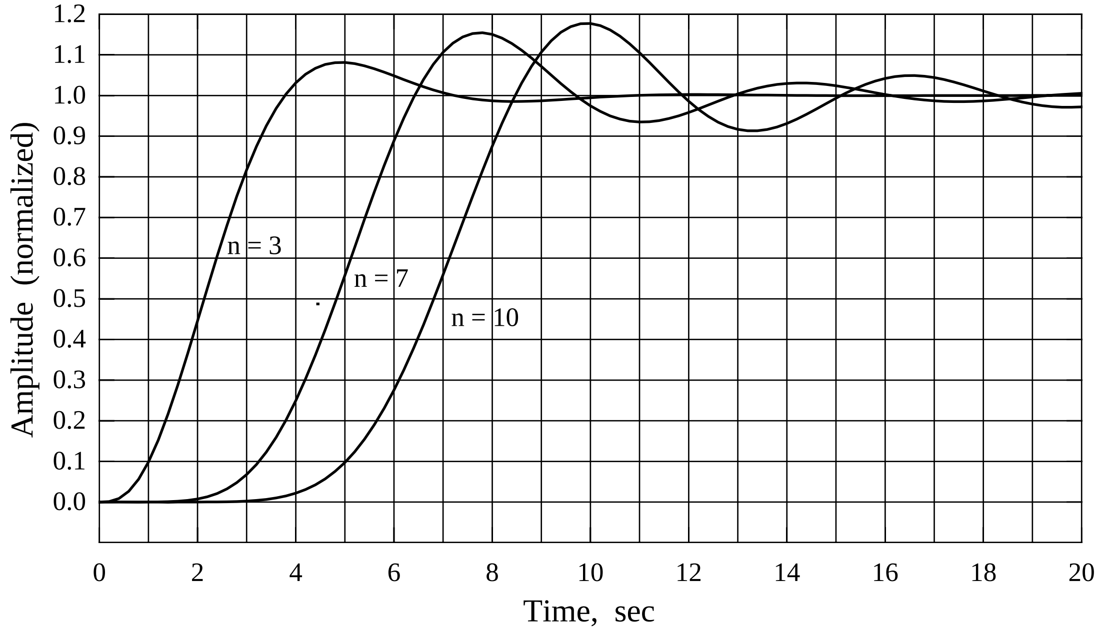
Overshoot is normally caused by an "excess" gain at high frequencies. By this we mean a low-pass frequency response with a peak value in the high frequency part of the pass-band.



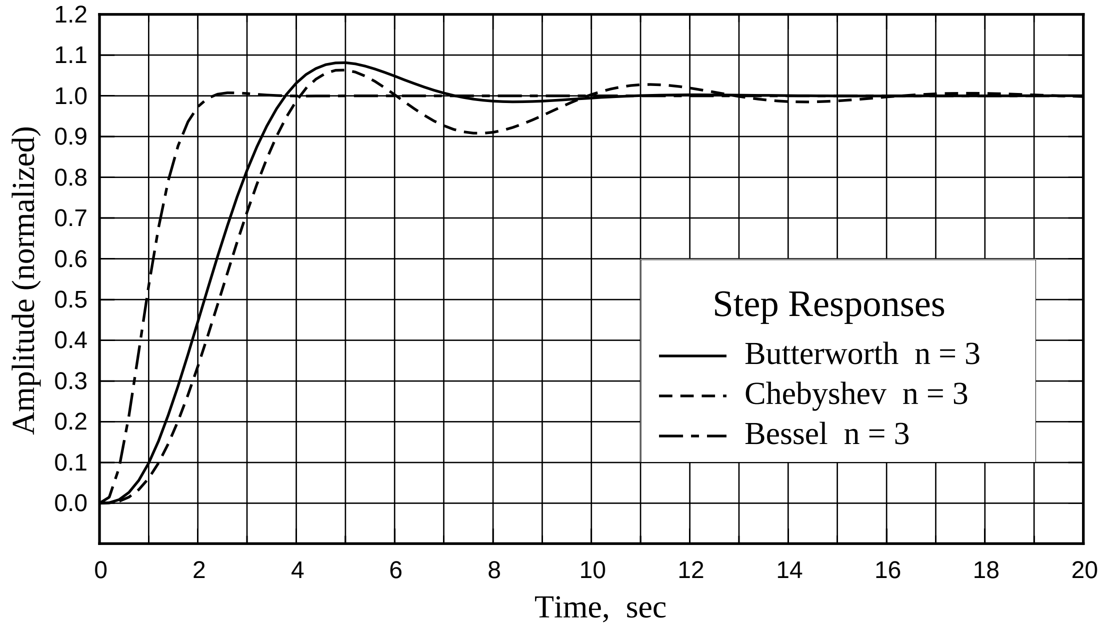
Comparison of a peaked response and a simple response from an RC filter.

The step responses of Butterworth filters of orders n = 3, 7 and 10 are shown in the next figure. Note that as n increases, the overshoot increases. This is because the higher order Butterworth filters have flatter magnitude characteristics (*i.e.* there is more gain at higher frequencies than in the lower order filters), although the response is never of the "peaked" type.

Ringing is due to sharp cutoff in the filter magnitude response and we can see that also increases when the order increases.



Step response of normalized Butterworth low-pass filters

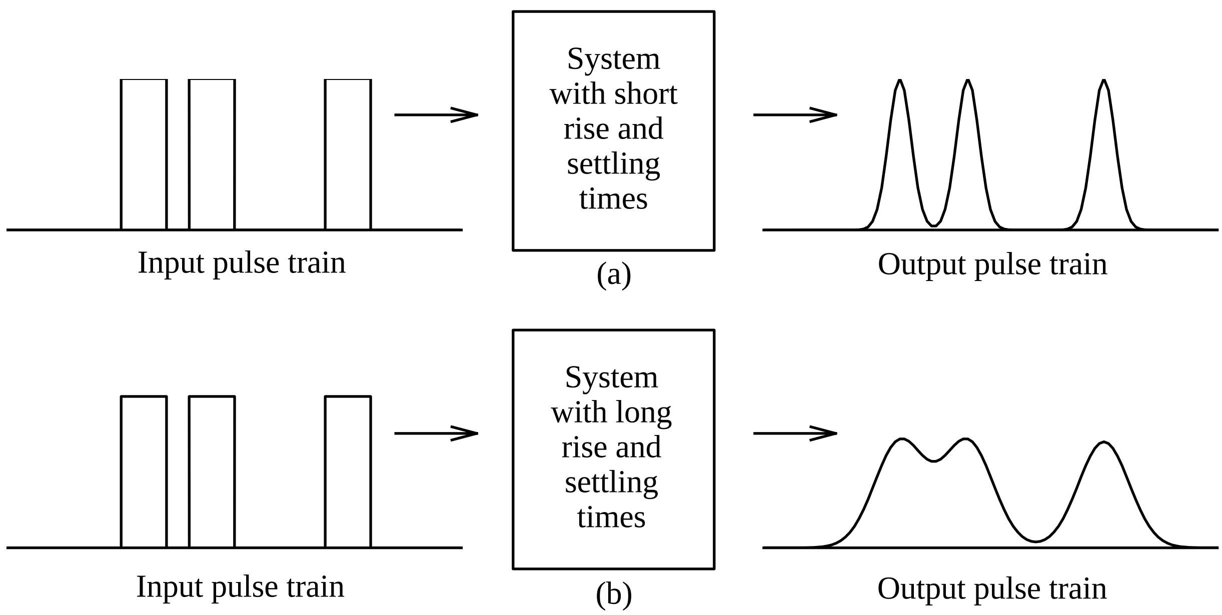


Comparison of filter transient response

The last figure compares the step response of a Bessel (linear phase) filter of order n = 3 to the response of an n = 3 Chebyshev filter with 1-dB ripple. Rise times cannot be compared because the bandwidths have not been adjusted to be equal. However, we can compare their ringing and settling times. The Chebyshev filter has a sharper cutoff, and therefore, more ringing and longer settling time than the Bessel filter. Note also that the overshoot of the Bessel filter is negligible. This is characteristic of this class of filters.

The decision of which filter is best depends strongly upon the particular situation. In certain applications, such as in the transmission of music, phase is not important. In these cases, the sharpness of the cutoff may be the dominant factor so that the Chebyshev or the Optimum filter are preferred to the others.

If we are dealing with pulse transmission instead, the requirement of the system is usually that the output sequence has approximately the same shape as the input sequence, except for a time delay of $T = T_2 - T_1$, as shown in the next figure. It is clear that a filter with a long rise time is not suitable, because the pulses would "smear" over each other as shown in the figure. The same can be said for long settling times. Since a pulse transmission system must have linear phase to ensure undistorted harmonic reconstruction at the receiver, the best filter for these systems is one with linear phase and small rise and settling times.

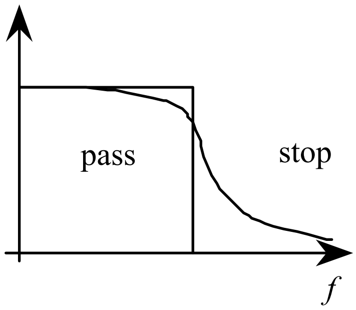


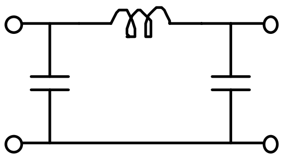
Smearing of pulses in systems with long rise and settling times.

Prototype filter networks

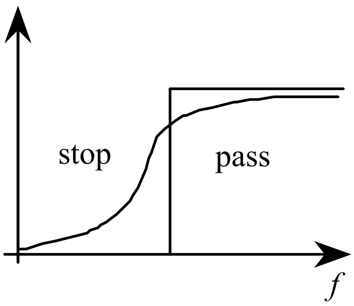
Typical frequency response and simple LC *realizations*

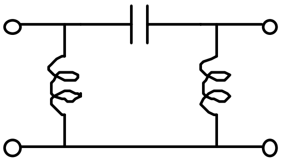
Low-pass



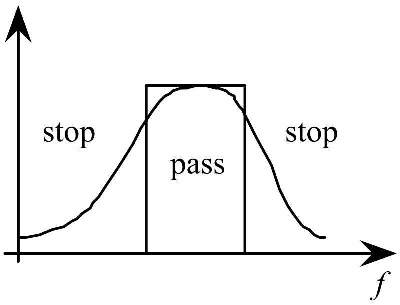


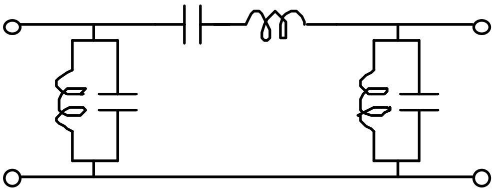
High-pass



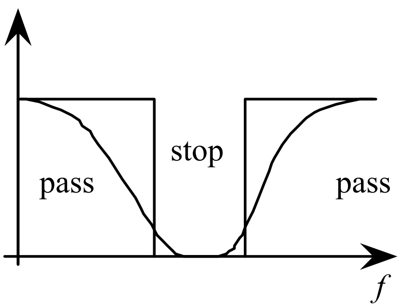


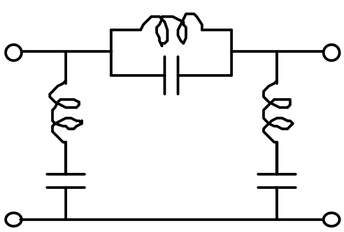
Band-pass





Band-stop





Two-port reactive circuit synthesis   Filter synthesis

From the transfer function to the circuit

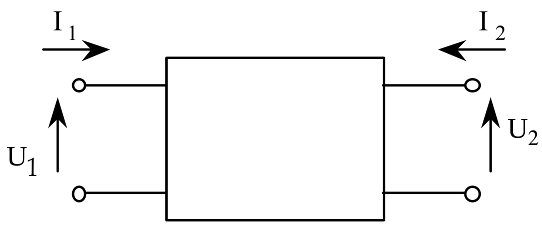
There are (at least) two ways of starting with some desired response and then, trying to realize it systematically.

1) The "Darlington procedure" starts with a desired insertion loss function, as a function of the frequency and then continues to find a reactive two-port to give this function.

2) The procedure we will follow is similar, but instead of the insertion loss function, it takes a ***transfer function***, effectively, the ratio of output to input voltage or current.

The impedance matrix Z is defined from:

$\boldsymbol{U = Z\,I}$ in matrix form



or in full:

$U_1 = Z_{11} I_1 + Z_{12} I_2$
$U_2 = Z_{21} I_1 + Z_{22} I_2$ from where:

$$Z_{11} = \left.\frac{U_1}{I_1}\right|_{I_2=0} \qquad Z_{12} = \left.\frac{U_1}{I_2}\right|_{I_1=0}$$

$$Z_{21} = \left.\frac{U_2}{I_1}\right|_{I_2=0} \qquad Z_{22} = \left.\frac{U_2}{I_2}\right|_{I_1=0}$$

Z_{11} and Z_{22} are driving point immittances, as they are concerned with a 1-port (with the other pair of terminals "open-circuited").

Z_{12} and Z_{21} are transfer impedances, being concerned with voltages and currents in different ports.

If one was concerned with filter design where the load was effectively an ***infinite impedance*** (open circuit), the above should be the starting equations; but two-ports are usually designed to give a specified response with a ***real, resistive load***.

If we connect a resistor of value R (a load) to the output terminals, the second of the impedance matrix equations now becomes:

$$U_2 = Z_{21} I_1 + Z_{22} I_2 = -R\, I_2$$

so that: $$CT(s) = \frac{I_2}{I_1} = \frac{-Z_{21}}{R + Z_{22}}$$

Starting now with the admittance matrix $I = Y\,U$, would similarly give:

$$UT(s) = \frac{U_2}{U_1} = \frac{-Y_{21}}{G + Y_{22}} \qquad \text{where } G = 1/R$$

Synthesis of I_2/I_1 or U_2/U_1 with a 1 Ohm load

We will now consider the synthesis of a reactive ladder network with a 1 Ohm termination to meet a specified voltage-transfer or current-transfer function; either

$$UT(s) = \frac{-Y_{21}}{1 + Y_{22}} \qquad \text{or} \qquad CT(s) = \frac{-Z_{21}}{1 + Z_{22}}$$

It can be demonstrated that as the numerators are transfer functions of reactive two-ports, they must be ***odd*** functions of the complex frequency s (although we will skip the proof), that is: $Z_{21}(s) = -Z_{21}(-s)$.

Secondly, we can remember (from page 20) that Z_{22} and Y_{22}, being d.p.i.'s are ratios of either ***odd to even*** or ***even to odd*** polynomials in s.

Suppose that one of the transfer functions above, say $CT(s)$, is given as the ratio of two polynomials in s, like:

$$CT(s) = \frac{P(s)}{E(s) + O(s)}$$

where the polynomial in the denominator has been split up into its even and odd parts, and $P(s)$ is either even or odd. The problem now is how to rearrange this equation into the form of the previous equation above.

The answer is to divide both numerator and denominator by either $E(s)$ or $O(s)$, the choice being such that the resultant numerator is an odd function of s, – which is necessary. In other words, we have to choose the part with the opposite parity to $P(s)$.

The denominator now becomes either: $1 + O(s)/E(s)$ or $1 + E(s)/O(s)$ which corresponds to the form of the denominator of the equation at the beginning. From this last expression we can easily separate the form of Z_{22}, which is the d.p.i. for a reactive circuit (the open-circuit output impedance on page 33) and so can be routinely realized.

Example

Suppose we want to realize the following current-transfer function:

$$CT(s) = \frac{2}{s^3 + 3s^2 + 4s + 2} = \frac{-Z_{21}}{1 + Z_{22}}$$

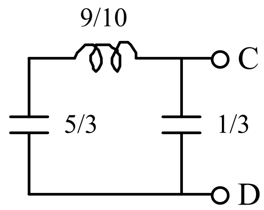
Comparing this with the previous equations, we see that $P(s)$ is even in s and so must be divided by the odd part of the denominator: $(s^3 + 4s)$.

Back to the first equation, we have:

$$Z_{21} = \frac{2}{s^3 + 4s} \qquad \text{and} \qquad Z_{22} = \frac{3s^2 + 2}{s^3 + 4s}$$

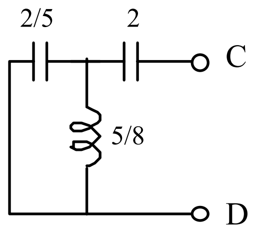
There is still one last problem to solve in order to realize this Z_{22}. Using the Euclid algorithm, there are various possibilities and then we may find more than one solution to this stage. (Note that since we are synthesizing Z_{22}, the first elements to emerge are those closest to the output terminals.

Z		**Y**
	$3s^2 + 2 \;:\; s^3 + 4s$	$s/3$
	$s^3 + 2s/3$	
$9s/10$	$3s^2 \leftarrow 10s/3$	
	$2 \rightarrow 10s/3$	$5s/3$
	0	

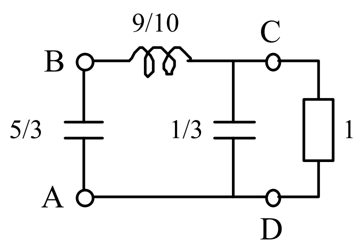


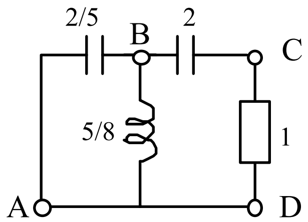
and

Z		**Y**
$1/(2s)$	$3s^2 + 2 \;:\; s^3 + 4s$	
	$s^2/2 + 2$	
	$5s^2/2 \rightarrow 4s$	$8/(5s)$
$5/(2s)$	$5s^2/2 \leftarrow s^3$	
	0	



Two possibilities seem to emerge, but these are the open-circuit impedances looking into the output terminals of the network of page 33, which should therefore look like:





Here we have attached the 1 Ohm resistive load to the C and D terminals of the circuits, so that we have indeed realized Z_{22}. Points A and B appear to be the only terminals left above to become the port 1 of the circuit in page 33.
The right hand circuit above is, in fact, a band-pass filter and the left-hand is low-pass.

The desired current-transfer function $CT(s)$ satisfies:

$$CT(s) \to 1 \text{ as } s \to 0, \qquad \text{and } CT(s) \to 0 \text{ as } s \to \infty$$

which is consistent only with the left-hand circuit above and not with the other.

So, in the end only one circuit emerges successfully from the two possibilities.

We now need to consider how to find sensible transfer functions from the filter specifications.

Obtaining a transfer function from a desired frequency response

A filter specification in the frequency domain usually consists of a specification of the power as a function of the frequency or equivalently, the amplitude response: $|V_2/V_1|(\omega)$ or $|I_2/I_1|(\omega)$. This specifies a ***real function of ω***, so the problem is, how to convert this into a ***complex function of s***, namely a transfer function $VT(s)$ or $CT(s)$.

Suppose that a desired response for a low-pass filter can be specified by:

$$\left|\frac{I_2}{I_1}\right|^2 = \frac{K}{1+P_n(\omega^2)} = |CT(j\omega)|^2 = h(-\omega^2) \qquad (1)$$

$P_n(\omega^2)$ is a polynomial of the form: $a_1\omega^2 + a_2\omega^4 + \ldots + a_n\omega^{2n}$, and K is the d.c. "gain". This clearly corresponds to a low-pass since all "zeros of transmission" are at $s = \infty$.

Now we want to find $CT(s)$. We first note that:

$$|CT(j\omega)|^2 = CT(j\omega)\cdot CT^*(j\omega) = CT(j\omega)\cdot CT(-j\omega)$$

since $CT(s)$ is the ratio of polynomials with real coefficients.

We can now define an extension of this function into the complex plane as:

$$h(s^2) = h(-\omega^2)\Big|_{-\omega^2 = s^2}$$

and, substituting $-\omega^2$ by s^2 in (1) we have:

$$CT(s)\cdot CT(-s) = h(s^2)$$

We observe that the roots of $CT(-s)$ are the negatives of those of $CT(s)$. Then, from $h(s^2)$ we can separate $CT(s)$ from $CT(-s)$ simply ***selecting the roots that lie in the left-hand half-plane***. This can be seen clearly with an example:

Butterworth Filters:

If we consider a Butterworth or maximally flat response, as many derivatives of $h(-\omega^2)$ must vanish at $\omega = 0$. This is equivalent to asking for as many derivatives as possible of $P(\omega^2)$ w.r.t. ω^2 to vanish at $\omega = 0$, and consequently, all $(n-1)$ first coefficients of P must be zero and the polynomial has the form:

$$P(\omega^2) = a_n\,\omega^{2n}$$ **Exercise:** Prove this.

The cut-off frequency (3-dB point) is then given by:

$$|CT(j\omega)|^2 = 1/2 = \frac{1}{1 + a_n \omega_c^{2n}}$$

And a value of $a_n = 1$ will give a normalized cut-off frequency of $\omega_c = 1$.

Finally, then: $|CT(j\omega)|^2 = 1/(1 + \omega^{2n}) = CT(j\omega) \cdot CT(-j\omega)$

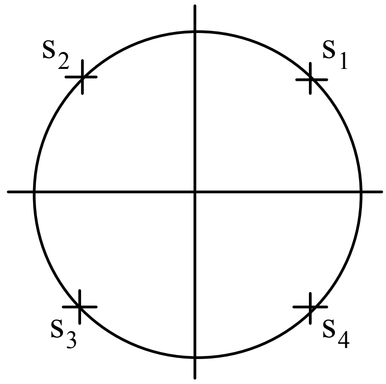
In particular, if we consider a second order filter with $n = 2$, so that: $CT(j\omega) \cdot CT(-j\omega) = 1/(1 + \omega^4)$, and using $-\omega^2 = s^2$:

$$CT(s) \cdot CT(-s) = 1/(1 + s^4)\ ,$$

the roots are all on the unitary circle and are symmetrically distributed with respect to the $j\omega$ axis.

We can separate then:

$$CT(s) = \frac{\pm 1}{(s - s_2)(s - s_3)}$$

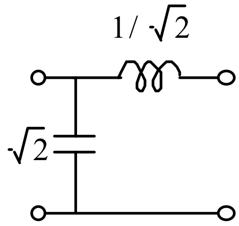


since these are the roots on the left-hand half-plane.

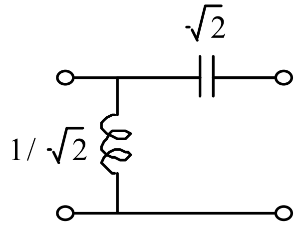
We now have:

$$CT(s) = \frac{\pm 1}{s^2 + \sqrt{2}s + 1}$$

On pages 34 to 36 we saw how to synthesize a CT function. For this example this will give:

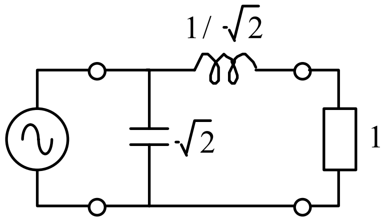


and



as the two possible realizations of: $Z_{22} = \dfrac{s^2 + 1}{\sqrt{2}\, s}$

The first is the network for a low-pass filter with response:
$CT(s) \cdot CT(-s) = \dfrac{1}{1 + s^4}$, so our second order Butterworth filter is as in the figure:



Similarly, for a third order Butterworth filter:

$$CT(j\omega)\cdot CT(-j\omega)=\frac{1}{1+\omega^6}=\frac{1}{1-(-\omega^2)^3}$$

$$CT(s)\cdot CT(-s)=\frac{1}{1-s^6}=\frac{1}{(1+2s+2s^2+s^3)(1-2s+2s^2-s^3)}$$

In general this factorised form can be written as:

$$CT(s)\cdot CT(-s)=\frac{1}{B_n(s)\cdot B_n(-s)} \qquad \text{and} \qquad CT(s)=\frac{1}{B_n(s)}$$

where $B_n(s)$ is called ***Butterworth polynomial*** of order n.

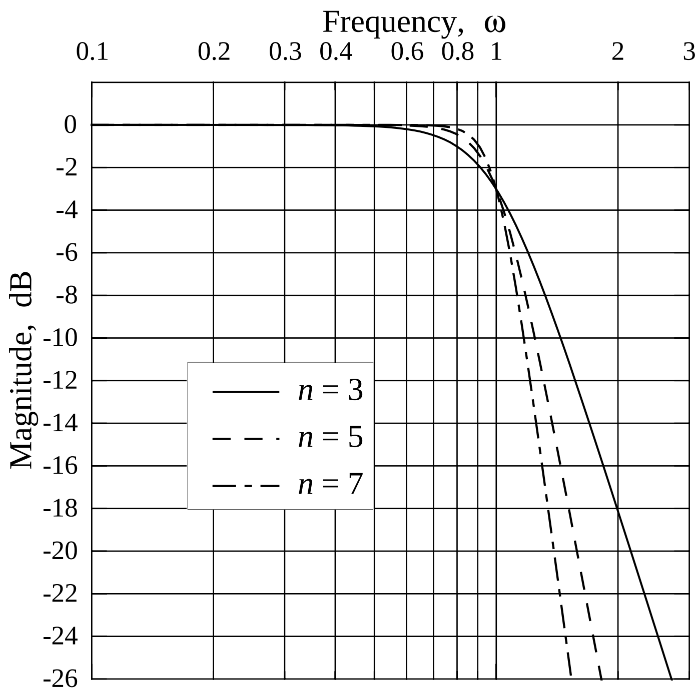
The synthesis approach used above can be applied to Butterworth filters of any order, and the adequate factorisation (the corresponding $B_n(s)$) is given in the following table:

$$B_n(s)=\sum_{i=0}^{n} a_i s^i \quad \text{with } a_0=1$$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1							
2	$\sqrt{2}$	1						
3	2	2	1					
4	2.613	3.414	2.613	1				
5	3.236	5.236	5.236	3.236	1			
6	3.864	7.464	9.141	7.464	3.864	1		
7	4.494	10.103	14.606	14.606	10.103	4.494	1	
8	5.126	13.138	21.848	25.691	21.848	13.138	5.126	1

The same polynomials can be written in factorised form to give directly the location of the roots:

n	Butterworth Polynomials (Factorised form)
1	$s+1$
2	$s^2+\sqrt{2}\,s+1$
3	$(s^2+s+1)(s+1)$
4	$(s^2+0.7653s+1)(s^2+1.84776s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$
6	$(s^2+0.5176s+1)(s^2+\sqrt{2}\,s+1)(s^2+1.9318s+1)$
7	$(s+1)(s^2+0.4450s+1)(s^2+1.2456s+1)(s^2+1.8022s+1)$
8	$(s^2+0.3986s+1)(s^2+1.1110s+1)(s^2+1.6630s+1)(s^2+1.9622s+1)$



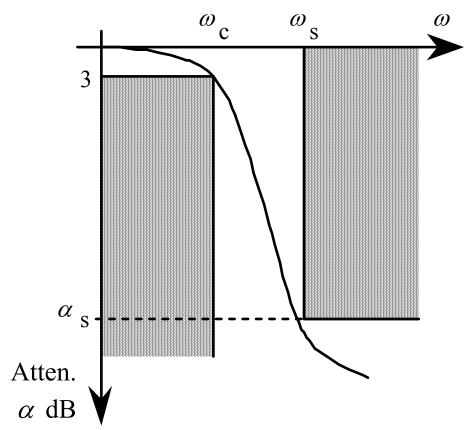
The figure shows the amplitude response of some of the Butterworth polynomials. Note that the response gradually approaches the ideal as the order increases.

The amplitude response curve shows how, with the 3-dB point at unit frequency, the curve approaches the rectangular "ideal", with the response falling off like $1/\omega^n$ as ω increases ($20n$ dB per decade or $6n$ dB per octave).

But the response to the step in page 34 show the price of this increasing "sharpness" in the frequency response. We see rise time, delay time, settling time, overshooting and ringing all getting worse.

Example:

Find an expression to calculate the minimum order n of a normalized Butterworth low-pass filter, to have an attenuation of at least α_s dB at a frequency ω_s outside the passband.



We can define the attenuation α as
$\alpha = -20 \log |T|$ dB.

(It is simply the negative of the amplitude in dB).

For a normalized Butterworth:

$$|T|^2 = \frac{1}{1+\omega^{2n}} = \frac{1}{1+(\omega/\omega_c)^{2n}}$$

then $\alpha(\omega) = 10 \log (1 + \omega^{2n})$

$$\therefore \quad \alpha/10 = \log (1 + \omega^{2n})$$

$$10^{\alpha/10} = 1 + \omega^{2n} \quad \Rightarrow \quad \omega^{2n} = 10^{\alpha/10} - 1 \quad \Rightarrow \quad 2n \log \omega = \log(10^{\alpha/10} - 1)$$

and finally,
$$n = \frac{\log(10^{\alpha_s/10} - 1)}{2 \log \omega_s}$$

If we select for example, α_s = 20 dB, and $\omega_s = 2$ ($\omega_s = 2\omega_c$)
n results:

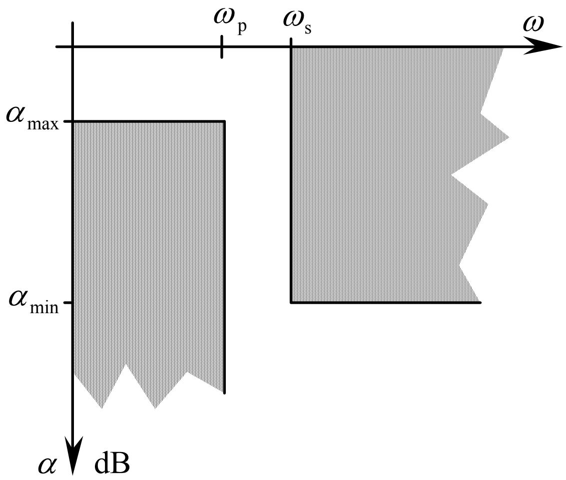
$$n = (\log 99)/(2 \log 2) = 3.3146 = 4$$

Problem 2

a) Consider the general form of a Butterworth frequency response:

$$|T(\omega)|^2 = \frac{1}{1 + (\omega / \omega_c)^{2n}}$$

Find an expression to calculate the minimum value of n (order of the filter) necessary to satisfy the following specifications:



where $\alpha = -20 \log_{10}|T|$ dB

α_{max} : Maximum attenuation allowed in passband.

α_{min} : Minimum attenuation required for $\omega > \omega_s$

(Note that the 3-dB cutoff frequency is not given)

b) Given the values α_{max} = 0.5 dB, $\omega_p = 10^5$ [rad/sec], α_{min} = 20 dB, $\omega_s = 2\times10^5$ [rad/sec], find the minimum value of n (approximated to an integer) to satisfy this specification. Choose the values given for ω_s and α_{min} to determine the value of the 3-dB cutoff frequency ω_c using the calculated value of the order of the filter.

Problem 3

Synthesize a normalized Butterworth or maximally flat filter of order 4 giving the value of all components.

Using this normalized prototype, design a low-pass filter with cutoff frequency of $\omega_c = 10^6$ rad/sec and terminated with a resistive load of 600Ω.

.

Chebyshev filters (equal ripple)

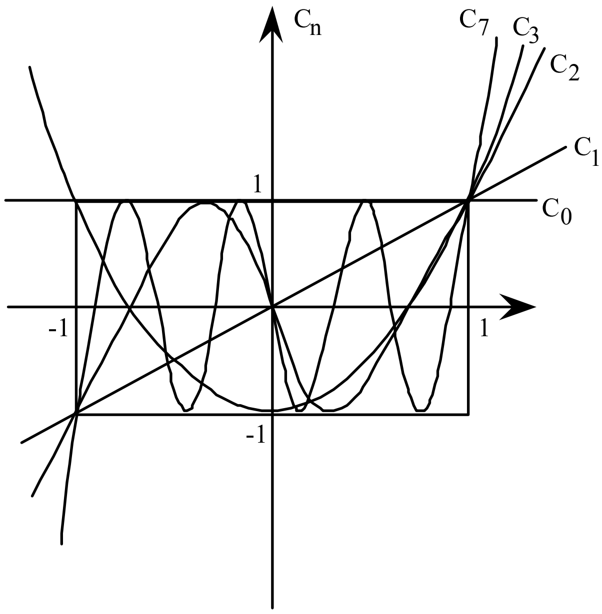
Similarly to the case of Butterworth filters, the Chebyshev response comes from:

$$CT(s) \cdot CT(-s) = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

where the $C_n(\omega)$ are the Chebyshev polynomials and ε is a parameter used to control the amount of ripple in the passband.

Let's first examine some of the properties of the Chebyshev polynomials:

- $|C_n(\omega)| \begin{cases} \leq 1 & |\omega| \leq 1 \\ > 1 & |\omega| > 1 \end{cases}$
- All zeros are in the interval [–1, 1].
- Maxima and minima are all in the same interval and have the same magnitude.
- For given n, the slope of the polynomial at $|\omega| > 1$ is the highest that can be obtained with any polynomial of the same order.



Some of the first polynomials are:

$C_0(\omega) = 1$
$C_1(\omega) = \omega$
$C_2(\omega) = 2\omega^2 - 1$
$C_3(\omega) = 4\omega^3 - 3\omega$
$C_4(\omega) = 8\omega^4 - 8\omega^2 + 1$
$C_5(\omega) = 16\omega^5 - 20\omega^3 + 5\omega$
$C_6(\omega) = 32\omega^6 - 48\omega^4 + 18\omega^2 - 1$
$C_7(\omega) = 64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$
$C_8(\omega) = 128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$
$C_9(\omega) = 256\omega^9 - 576\omega^7 + 432\omega^5 - 120\omega^3 + 9\omega$
$C_{10}(\omega) = 512\omega^{10} - 1280\omega^8 + 1120\omega^6 - 400\omega^4 + 50\omega^2 - 1$

In general we can write:

$C_n(\omega) = \cos(n \cos^{-1}\omega)$

How a Chebyshev polynomial can be used to form a filter response?

Consider for example the third order polynomial C_3:

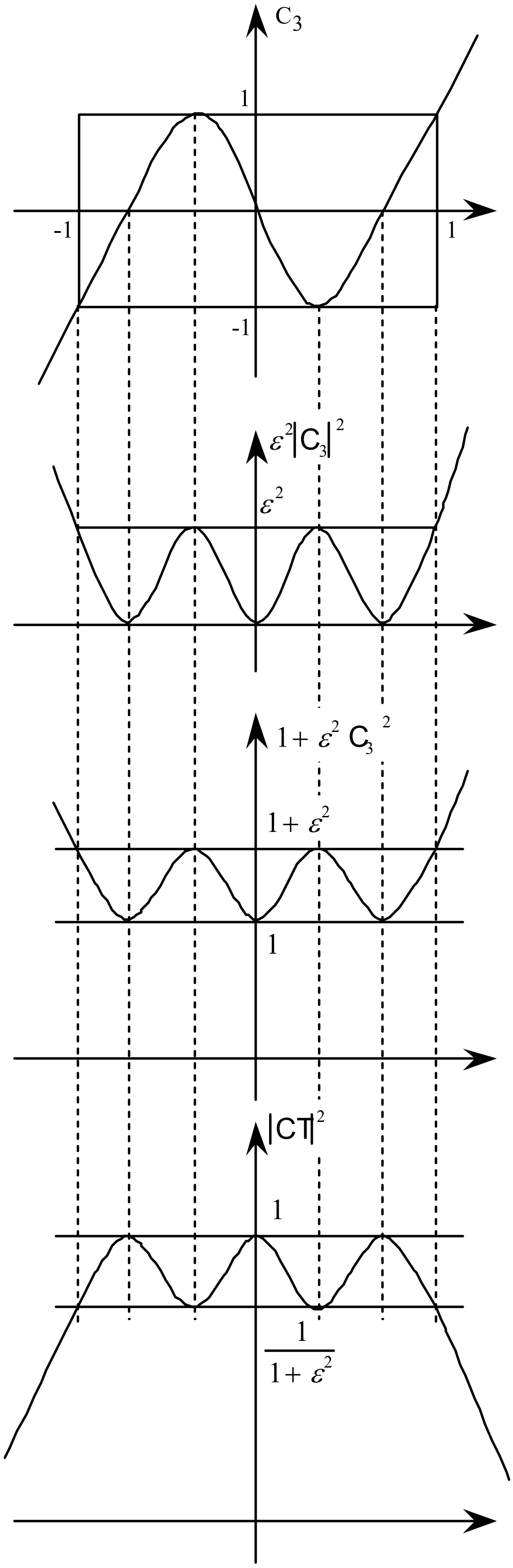
Now, we make: $\varepsilon^2 |C_3|^2$

If we now form:

$1+\varepsilon^2 |C_3|^2$

And finally we get:

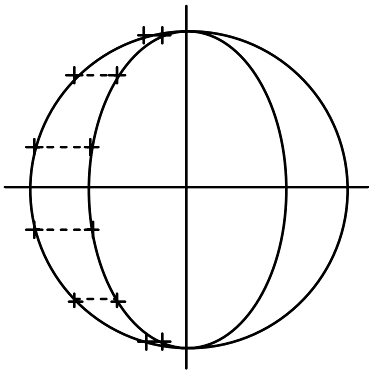
$|CT|^2 = \dfrac{1}{1+\varepsilon^2 |C_3|^2}$



We saw in page 37 that the poles in the Butterworth response were all located on the unitary circle. In the case of Chebyshev filters, the poles are on an ellipse as shown in the figure. There we can compare the position of the roots for Butterworth and Chebyshev responses of order 6.

The actual shape of the ellipse depends on the amount of ripple (ε).

The smaller the ripple, the nearest is the ellipse to the circle.



Problem 4

Find a passive ladder circuit realization of a low-pass Chebyshev filter of order five with 0.5 dB ripple in the passband. Scale component values to give a cutoff frequency ω_c of 10^6 [rad/sec] (f_c = $1/(2\pi)$ MHz) and terminated with a resistive impedance load of 600 Ohm.

Often, in the design process it is convenient to write the denominator polynomial of the transfer function *CT(s)* in a factorised form.

The corresponding factorisation of the denominator *D(s)* of *CT(s)* for Chebyshev filters is given in the following table for two different amounts of ripple.

$$CT(s) = \frac{1}{D(s)} \qquad D(s) = \begin{cases} \prod_{i=1}^{n/2} (s^2 + b_{1i}s + b_{0i}) & n = 2, 4, 6, \cdots \\ (s + b_1) \prod_{i=1}^{(n-1)/2} (s^2 + b_{1i}s + b_{0i}) & n = 3, 5, \cdots \end{cases}$$

Note: The symbol Π means multiple product.

Approximation	Order n	Factorised form: coefficients of $D(s)$ b_1	b_{11}	b_{01}	b_{12}	b_{02}	b_{13}	b_{03}
	2		1.414214	1				
Maximally flat	3	1	1	1				
(Butterworth)	4		0.765367	1	1.847759	1		
	5	1	0.618034	1	1.618034			
	6		0.517638	1	1.414214	1	1.931852	1
	2		1.425624	1.516203				
0.5 dB	3	0.626457	0.626456	1.142448				
equal-ripple	4		0.350706	1.063519	0.846680	0.356412		
(Chebyshev)	5	0.362320	0.223926	1.035784	0.586245	0.476767		
	6		0.155300	1.023023	0.424288	0.590136	0.579588	0.156997
	2		1.097734	1.102510				
1 dB	3	0.494171	0.494171	0.994205				
equal-ripple	4		0.279072	0.986505	0.673739	0.279398		
(Chebyshev)	5	0.289493	0.178917	0.988315	0.468410	0.429298		
	6		0.124362	0.990733	0.339763	0.557720	0.464125	0.124707

Frequency transformations

As mentioned before, a low-pass prototype can be converted to any other type of frequency response using an appropriate frequency transformation.

We already introduced frequency scaling by which a frequency ω is transformed to a new frequency ω' by:

$$\omega' = k_f\,\omega$$

and then:

$$\omega' L' = \omega L = (k_f\omega)(L/k_f) \qquad \therefore\quad L' = L/k_f$$
$$\omega' C' = \omega C = (k_f\omega)(C/k_f) \qquad \therefore\quad C' = C/k_f \tag{1}$$
$$R' = R$$

For example, the impedance of the inductor L' at the new frequency ω' is the same as that of the inductor L at the old frequency ω.

If we write in general the old frequency as a transformation of the new one: $\omega = T(\omega')$ or $s = T(s')$, for what functions $T(s')$ can the corresponding relations still apply?

The old impedances are:

$$sL = T(s')\,L$$
$$s\,C = T(s')\,C \tag{2}$$
$$R = R'$$

We can see that the only condition is that $T(s')L$ and $T(s')C$, or indeed simply $T(s')$, must be realizable as a d.p.i.

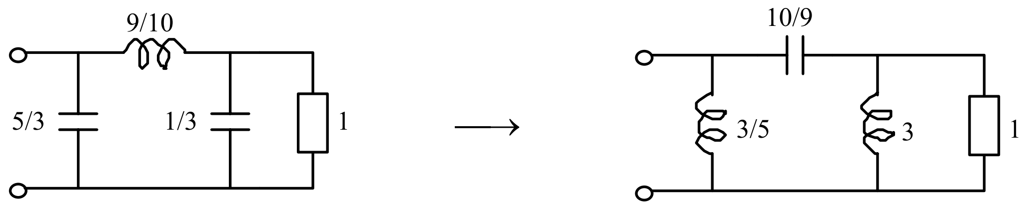
Transformation from low-pass to high-pass

Consider the transformation $s' = 1/s$ or $s = 1/s' = T(s')$
From equations (2), the impedance of the old elements should be now:

$$sL = (1/s')\,L = 1/s'C_1$$
$$sC = (1/s')\,C = 1/s'L_1$$
$$R = R'$$

Then, an inductor of value L is transformed to a capacitor of value $C_1 = 1/L$, and similarly: $L_1 = 1/C$ and all resistors remain unchanged.

For example the low-pass filter calculated in page 35 transforms to the high-pass version:



The circuit on the left (the low-pass filter from page 35) has identical insertion loss, and $|I_2/I_1|$ at frequency ω as the circuit on the right at frequency $1/\omega$.

Note that one can follow the $s' = 1/s$ transformation by a frequency scaling, or do it in one step by the transformation: $s' = K/s$. (or $s = T(s') = K/s'$)

Low-pass to band-pass transformation

A transformation $T(s')$: $s' \to s$ is possible providing that $T(s')$ represents a realizable immittance.

We know that the frequency scaling, $s' = as$ ($T(s') = s'/a$) is realizable as it is also the low-pass to high-pass $s' = b/s$ (or $s = T(s') = b/s'$). Clearly, a combination of both is also realizable, so we can examine:

$$s = T(s') = \frac{s'}{a} + \frac{b}{s'} \qquad \text{and}$$

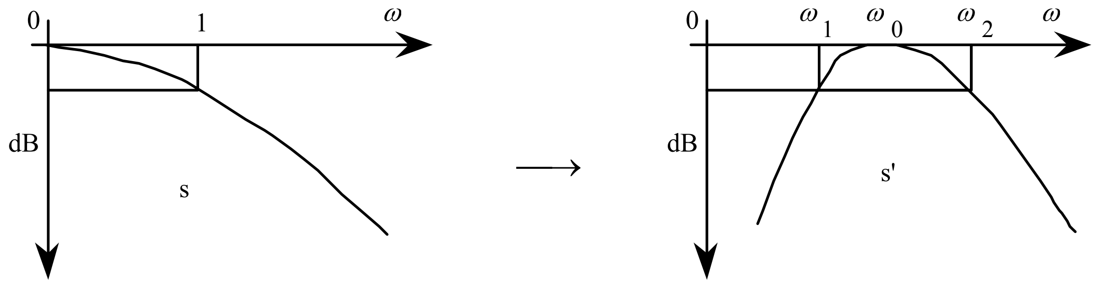
$$s = T(s') = 1\Big/\left(\frac{s'}{a} + \frac{b}{s'}\right)$$

We can re-write the first transformation as:

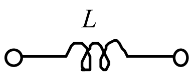
$$s = T(s') = \frac{\omega_0}{\Delta\omega}\left[\frac{s'}{\omega_0} + \frac{\omega_0}{s'}\right] \quad \text{where} \quad \Delta\omega = \omega_2 - \omega_1$$

and $\quad \omega_2/\omega_0 = \omega_0/\omega_1, \quad$ giving: $\quad \omega_0^{\,2} = \omega_1\omega_2.$

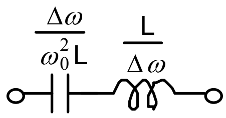
Using this transformation, a low-pass frequency response will transform as:



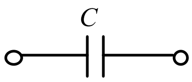
The individual elements will be transformed as:



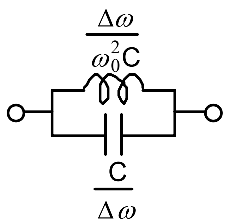
$$sL = T(s')L = \frac{s'\,L}{\Delta\omega} + \frac{\omega_0^2 L}{s'\,\Delta\omega}$$



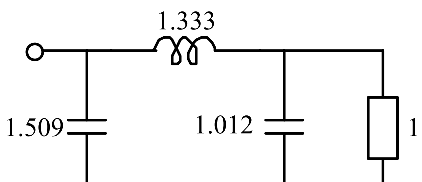
and



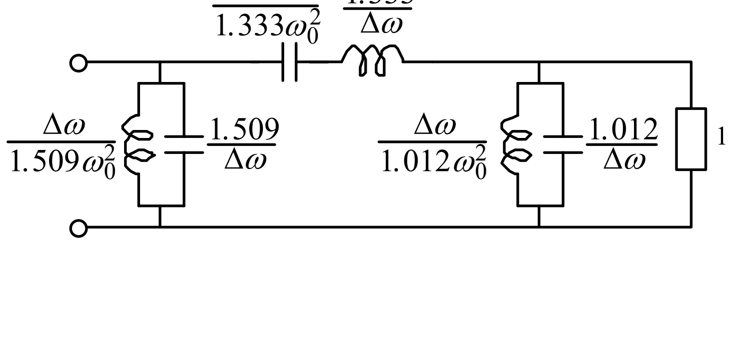
$$sC = T(s')C = \frac{s'\,C}{\Delta\omega} + \frac{\omega_0^2 C}{s'\,\Delta\omega}$$



In this way, the prototype low-pass filter of Chebyshev response (for 1 dB ripple): (***Exercise:*** Find the corresponding element values)



would transform to:



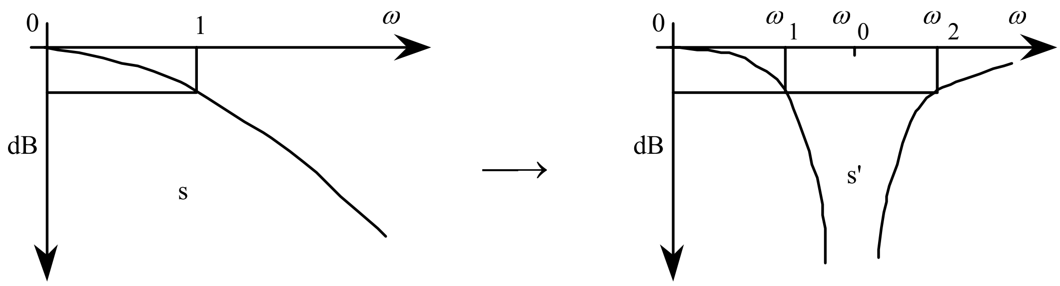
This is a Chebyshev filter with band-pass response. If we consider $\omega = \omega_0$ all *LC* pairs are at resonance giving perfect transmission, – corresponding to d.c. in the low-pass filter.

Low-pass to band-stop transformation

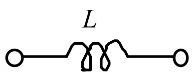
Finally, from the second expression in last page we get a low-pass to band-stop transformation. (The formula is the reciprocal of the previous one).

$s = T(s') = \dfrac{\Delta\omega}{\omega_0\left[\dfrac{s'}{\omega_0} + \dfrac{\omega_0}{s'}\right]}$ where again $\Delta\omega = \omega_2 - \omega_1$ and $\omega_0^{\,2} = \omega_1\omega_2$.

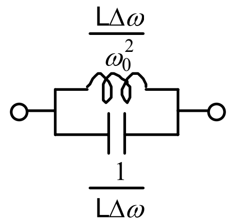
The low-pass and the corresponding transformed response are:



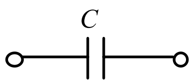
The individual elements will be transformed as:



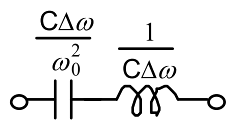
$$sL = T(s')L = \frac{s'\,\Delta\omega}{(s'^{\,2} + \omega_0^2)}\,L$$



and



$$sC = T(s')C = \frac{s'\,\Delta\omega}{(s'^{\,2} + \omega_0^2)}\,C$$



Richard's transformation

Let's now try the transformation: $s = T(s') = \tanh(s')$

or using a normalizing factor ω_0: $s = \tanh(s'/\omega_0)$

The question now is: what circuit component has a d.p.i. of the form:

$$\begin{aligned} s &= \tanh(s'/\omega_0) \\ &= \tanh(j\omega'/\omega_0) = j\tan(\omega'/\omega_0) \end{aligned}$$

The impedance of a piece of transmission line of electric length θ with characteristic impedance Z_c and terminated with a load Z_l is:

$$Z_c \frac{Z_l + jZ_c \tan\theta}{Z_c + jZ_l \tan\theta}$$

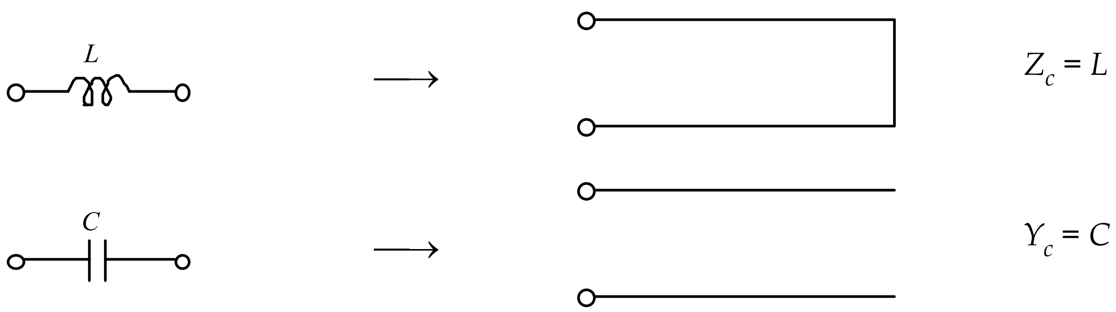
Then, a short-circuited piece with Z_c = 1 and length: $\theta = \beta l = \omega'/\omega_0$ has the impedance:

$$j\tan(\omega'/\omega_0)$$

Now, what component has an admittance of $j\tan\theta$?
It is easy to see that this corresponds to an ***open-circuited*** piece of transmission line of the same length l and characteristic impedance Z_c.

So, with this transformation, the circuit components will transform:

$$sL = jL\tan(\omega'/\omega_0) \qquad \text{(short circuit, } Z_c = L)$$
$$sC = jC\tan(\omega'/\omega_0) \qquad \text{(open circuit, } Y_c = C)$$

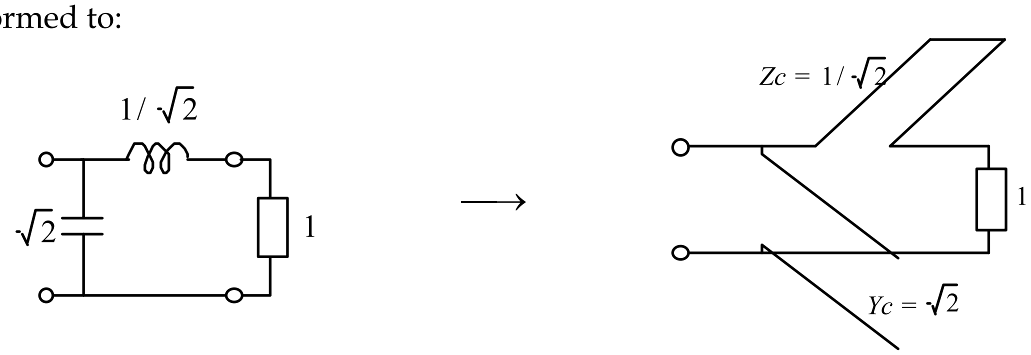


In both cases the electric length is $\theta = \omega'/\omega_0$

A network with a number of components $L_1, L_2, \ldots$ and $C_1, C_2, \ldots$ transforms to short-circuited lines with $Z_c = L_1, L_2, \ldots$ and open-circuited lines with $Y_c = C_1, C_2, \ldots$ simultaneously provided that ω_0 is the same for all lines.

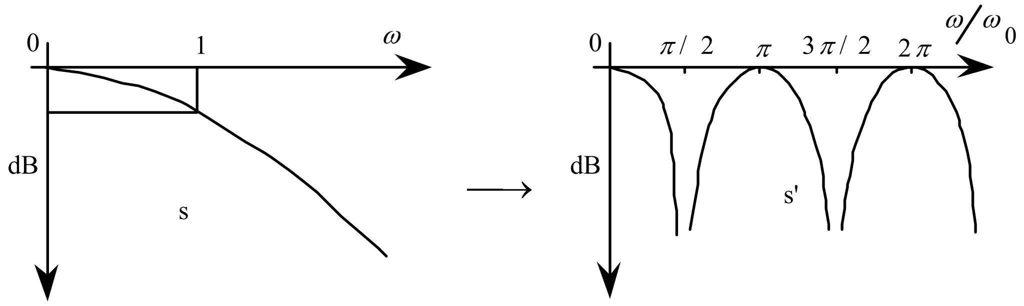
This transformation results in a ***periodic mapping***, since $\tan(\theta)$ is periodic with period π. For non-dispersive lines, where electric length is proportional to frequency (as assumed in the equations above), this means the transformation is periodic in frequency, the period being the frequency at which the line length is $\lambda/2$.

Using this transformation, the simple second order Butterworth filter of page 37 is transformed to:



Both pieces of transmission lines or "stubs" have the same Z_c in this case, – of 0.707 Ohm or 70.7 Ohm if we want a load of 100 Ohm.

The curves showing the corresponding transformation of the frequency response are:



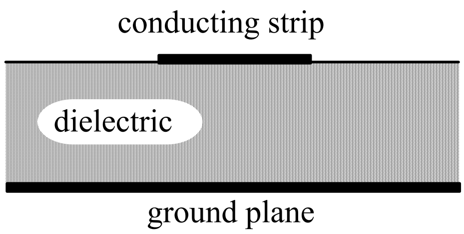
The resulting filter looks dual-purpose, in containing both band-pass and band-stop regions; but the maximally flat characteristic at d.c. of the low-pass prototype strictly transforms to $\omega'/\omega_0 = \pi, 2\pi, 3\pi$, etc.

This transformation is widely used at microwave frequencies, –say, 0.5 to 30 GHz.

The circuit above is not very convenient as it is and in practice some other transformations are used (Kurodo transformations and "unit elements") so that only shunt elements are needed (because that is more convenient in planar microwave circuits).

This transformation allows us to add transmission lines to the set of possible elements for realizing passive filters, –that is, to *L*, *C*, and *R*.

These filters are very easy to make in microstrip, – a sort of printed-circuit board for use at microwave frequencies.



For instance, a mask necessary for a 3-element filter would look like:

The 3 elements are shunt open-circuit stubs.

The 3 free variables allow us to choose the desired third order response (Chebyshev, Butterworth, or any other).

Problem 5

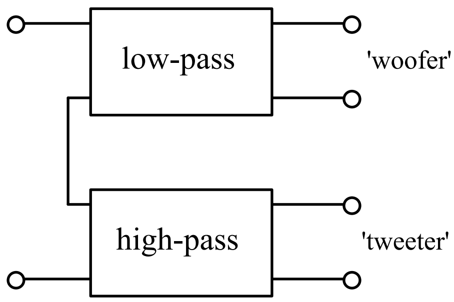
Starting from the Butterworth approximation for $|CT(j\omega)|^2$, find the function CT(s) corresponding to a third order low-pass filter and realize it as a ladder circuit.

Apply the low-pass to high-pass transformation to find a normalized third order high-pass filter.

These could be used as part of a 2-loudspeaker design as in the figure.

Verify that the impedance looking into the two circuits in series is 1 Ω, for all frequencies.

Finally, scale the two filters to provide: loudspeakers of 8 Ω and a cross-over frequency of $5000/\pi$ Hz. The latter, taken as cutoff frequencies, would provide 3 dB attenuation to each speaker.



Problem 6

A low-pass system function employing a third order Bessel polynomial (not normalized to 3dB cutoff at $\omega_c = 1$) is:

$$H(s) = \frac{15}{s^3 + 6s^2 + 15s + 15}$$

Find the corresponding frequency scaling to give a 3dB cutoff frequency of 1 rad/sec.

For this low-pass prototype, find a ladder realization and apply the necessary transformations to get a band-pass filter with $\omega_1 = 2\times10^4$ rad/sec, $\omega_2 = 8\times10^4$ rad/sec and a resistive load of 600 Ω.

Note: The 3-dB cutoff frequency of the transfer function above is $\omega_c = 1.75568$